Dirac–Nijenhuis structures

2004 J. Phys. A: Math. Gen. 37 7267

(http://iopscience.iop.org/0305-4470/37/29/007)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **37** (2004) 7267–7296 PII: S0305-4470(04)78049-X

Dirac–Nijenhuis structures

J Clemente-Gallardo and J M Nunes da Costa

Departamento de Matematica, Universidade de Coimbra, Apartado 3008,
3001-454 Coimbra, Portugal

E-mail: jesus@mat.uc.pt and jmcosta@mat.uc.pt

Received 17 March 2004, in final form 3 June 2004
Published 7 July 2004
Online at stacks.iop.org/JPhysA/37/7267
doi:10.1088/0305-4470/37/29/007

Abstract

In this paper we study the concept of Dirac–Nijenhuis structures. We consider them to be a pair $(D, \mathcal{N})$ where D is a Dirac structure, defined with respect to a Lie bialgebroid (A, A^*), and $\mathcal{N}$ is a Nijenhuis operator which defines a deformation of the Lie algebroid structure of D in a compatible way. The transformation can be considered to deform also the double of the Lie bialgebroid, which leads to the concept of Dirac–Nijenhuis structures of type I, or to not affect it, leading to Dirac–Nijenhuis structures of type II. We prove that the concept contains Poisson–Nijenhuis structures as a particular case and provides another example for the case of Kähler manifolds.

PACS numbers: 02.10.De, 02.20.Sv, 02.40.Ma
Mathematics Subject Classification: 58H15, 58H05, 53D17, 17B62

1. Introduction

Dirac structures were introduced by Courant and Weinstein and Dorfman at the end of the eighties [3, 4]. They are a generalization of Poisson structures which, roughly speaking, replace the canonical symplectic foliation of the Poisson manifold by a presymplectic one. Roughly speaking still, we can think of Dirac structures as a Poisson manifold endowed with a distinguished distribution which, speaking in mechanical terms, defines a set of implicit constraints. Mechanical systems with constraints, singular Lagrangian systems and many engineering systems are naturally described by a Dirac manifold. These applications in engineering are defined through the theory of port controlled Hamiltonian systems introduced by van der Schaft, Maschke and co-workers (see for instance [17, 22] and references therein).

From a geometrical point of view, Dirac structures are intimately related to Lie algebroids and Lie bialgebroids [2, 13, 12]. A Dirac structure on a manifold M was defined in [3, 2] as a subbundle D of the Whitney sum $TM \oplus T^*M$ satisfying certain properties, which correspond to the definition of a Lie algebroid structure. Later, the concept was generalized to similar subbundles defined on Whitney sums of the form $A \oplus A^*$, where (A, A^*) is a Lie bialgebroid [13].

On the other hand, the deformation of structures by using Nijenhuis operators is a concept often used in the literature. Originally proposed within the framework of integrable systems (see the introduction and references of [16]), it allows a deformation of Lie algebra structures defined on different types of manifolds. It has recently been extended to the Lie algebroid case, and therefore a very interesting example seems to be the study of the deformation of Lie algebroid structure which corresponds to a Dirac manifold. In [5], the problem was discussed for the case of Poisson manifolds (corresponding to the case of Poisson–Nijenhuis manifolds [9–11]). Within the Lie algebroid domain, the Jacobi–Nijenhuis case (i.e. the deformation associated with a Jacobi manifold) was also studied in [8, 18]. Very recently, two new works have been presented concerning more general settings:

- In [14], the concept of Dirac–Nijenhuis structures was introduced for the particular case of Dirac structures defined in the Whitney sum $TM \oplus T^*M$ for M a differentiable manifold. The tools used are similar to ours, but our approach is more general. There is an important property of the construction which is worth mentioning: the new Lie structure of the Dirac structure is not obtained from a trivial deformation of its Lie algebroid structure. The authors define a certain transformation of the double which yields a new Lie product on the Dirac structure, but it does not correspond to a Nijenhuis operator for the Lie algebroid structure.
- In [1], is studied the deformation of Courant algebroids, Lie bialgebroids and Dirac structures, by studying the deformation of Leibniz algebras. It is an interesting line of work, and it proves to be very powerful for the study of the deformation of Courant algebroids (the main objective of [1]), but the authors do not explore extensively the case of Dirac structures, and after defining the concept at a general level, they study only some simple cases, namely the graph of a presymplectic tensor or of a Poisson tensor on the double of the Lie bialgebroid $TM \oplus T^*M$. In these examples, we find the same situation as in the previous case: the new structure does not arise as a trivial deformation of the Lie algebroid structure of the Dirac structure. In this case, it is the effect of the deformation of the Leibniz algebra on the corresponding subbundle.

In this work, we study extensively the case of Dirac structures defined on the double of general Lie bialgebroids. We propose an extension to general Dirac structures, which includes the Poisson–Nijenhuis case as a particular example, and can be adapted to the Jacobi–Nijenhuis framework (see [19]). Our approach shares elements with [1] and with [14], but it also contains important differences, basically the fact that we consider deformations of the Lie algebroid structure of the Dirac bundle by defining Nijenhuis operators for them. Hence, we define the deformation of Dirac structures by deforming the skew-symmetric operation which defines a Lie bracket on the Dirac bundle, as the authors of [14], but we do it in a more general setting, namely Dirac structures defined on arbitrary Lie bialgebroids. Besides, we allow the transformation to map the Dirac bundle on itself, a property which is not required in [14].

Cariñena *et al* [1] also study the general Lie bialgebroid case, but using very different tools, namely the deformation of Leibniz algebras. Both approaches are related to each other, and some of our results concerning the deformation of the double of a Lie bialgebroid can be obtained from the results in [1], but the philosophies of work are quite different, since this work focuses on the deformation of general Dirac structures, while in [1] that is defined as a side result (the main goal being the deformation of Courant algebroids), and not developed in detail. Besides, the deformations provided in these examples do not define trivial deformations of the Lie algebroid structure of the Dirac bundle.

Therefore, we consider that our work offers a new approach regarding the deformation of Dirac structures. In this paper, we will consider only deformations of the double which arise from a deformation of the Lie bialgebroid, and not more general ones, defined directly at the level of the double, which will be studied in a future paper.

The structure of the paper is as follows. In section 2 the notion of Dirac structure in general Lie bialgebroids is presented: we discuss the definition of a Lie bialgebroid, and some properties and examples, as well as the notion of Courant algebroids. Then, Dirac structures are defined as suitable subbundles within that framework. The notion of the characteristic pair of a Lie algebroid, which will be very important for us later, is also discussed. The remaining sections are devoted to the deformation of the structures presented so far: section 3 studies the deformation of Lie algebroids and Lie bialgebroids, since they are necessary to define the deformation of Dirac structures themselves. Then, section 4 studies some simple cases of deformations of the double of a Lie bialgebroid. Finally, we arrive at our main interest, the deformation of Dirac structures. We study the problem following two different approaches: in section 5 we study the deformations of Dirac structure arising from the deformations of the double of the Lie bialgebroid where they are defined. In section 6 we study the second framework: a deformation of the Lie algebroid structure of the Dirac structure which is defined only for D, and not for the double. Finally, section 7 introduces the notions of Dirac–Nijenhuis structures of types I and II and provides examples of both cases. Besides, we compare carefully our definitions with the definitions presented in [1, 14].

2. Preliminaries: Dirac structures on Lie bialgebroids

Given a Lie bialgebroid (A, A^*), we can consider the Whitney sum $B \equiv A \oplus A^*$, the duality between the two bundles can be used to define two canonical forms, one symmetric $(\cdot,\cdot)_+ : B \times B \to \mathbb{R}$ and one skew-symmetric $(\cdot,\cdot)_- : B \times B \to \mathbb{R}$:

$$((X_1, \alpha_1), (X_2, \alpha_2))_\pm = \langle X_2, \alpha_1\rangle \pm \langle X_1, \alpha_2\rangle \qquad \forall (X_1, \alpha_1), (X_2, \alpha_2) \in B. \tag{1}$$

We can also define a skew-symmetric operation on $A \oplus A^*$:

Definition 2.1. *Consider the Whitney sum bundle $B = A \oplus A^*$. We can endow the set of sections of B with a bilinear, skew-symmetric operation, in the form:*

$$[(X_1, \alpha_1), (X_2, \alpha_2)]_\diamond = ([X_1, X_2] + [(X_1, \alpha_1), (X_2, \alpha_2)]_{\mathcal{L}^*}, [(X_1, \alpha_1), (X_2, \alpha_2)]_{\mathcal{L}} + [\alpha_1, \alpha_2]_*) \tag{2}$$

with

$$[(X_1, \alpha_1), (X_2, \alpha_2)]_{\mathcal{L}^*} = \mathcal{L}^*_{\alpha_1} X_2 - \mathcal{L}^*_{\alpha_2} X_1 - \tfrac{1}{2} d_*\big(i_{X_2}\alpha_1 - i_{X_1}\alpha_2\big)$$

and

$$[(X_1, \alpha_1), (X_2, \alpha_2)]_{\mathcal{L}} = \mathcal{L}_{X_1}\alpha_2 - \mathcal{L}_{X_2}\alpha_1 + \tfrac{1}{2} d\big(i_{X_2}\alpha_1 - i_{X_1}\alpha_2\big)$$

where $\mathcal{L}_X$, $\mathcal{L}^*_\alpha$ correspond to the Lie derivatives on A and A^*.

For this operation, bi-linearity and skew-symmetry are trivial to prove. On the other hand, Jacobi identity is not satisfied on the whole space of sections of B. It is possible, though, that the property holds for the space of sections of a subbundle $D \subset B$.

Definition 2.2. *A subbundle $D \subset B$ which is maximally isotropic with respect to the operation $(\cdot,\cdot)_+$ in (1) is said to be a* Dirac structure *if the skew-symmetric bracket (2) defines a Lie bracket on the space of sections ΓD.*

It is also well known that

Proposition 2.1. *Consider the Whitney sum bundle* $B = A \oplus A^*$ *for the Lie bialgebroid* (A, A^*). *Consider also the operation* $[\cdot,\cdot]_\diamond$ *and the mapping* $\rho_B = \rho \oplus \rho_* : B \to TM$. *A subbundle* $D \subset B$ *is a Dirac structure if and only if* $(D, [\cdot,\cdot]_\diamond, \rho_B|_D)$ *is a Lie algebroid.*

2.1. Characteristic pairs

The characterization of Dirac structures can be done in terms of subbundles of A and suitable A-tensors. This generalizes the description proposed in [22] for the case of Dirac structures defined on $TM \oplus T^*M$. We will follow Liu's construction, described in [12]. For the sake of simplicity, we will restrict ourselves to the case where the intersection $D \cap A$ has constant rank and defines a subbundle of the bundle A.

Definition 2.3. *Consider a Lie bialgebroid* (A, A^*) *and a maximally isotropic subbundle of its Whitney sum* $D \subset A \oplus A^*$. *Any pair of a smooth subbundle* $I \subset A$ *and a bivector* $\Omega \in \Gamma\left(\bigwedge^2 A\right)$ *corresponds to a maximally isotropic subbundle of* $A \oplus A^*$ *with respect to the symmetric product in (1). The* characteristic pair *of the Dirac structure* D *is a pair* (I, Ω) *which corresponds to it. The subbundle* $I \subset A$ *is called the* characteristic bundle. *The expression of the Dirac structure is as follows:*

$$D = \{(X + \Omega^\# \alpha, \alpha) | \forall X \in I, \forall \alpha \in I^\perp\} = I \oplus \mathrm{graph}(\Omega^\# | I^\perp) \tag{3}$$

where $I^\perp \subset A^*$ *stands for the co-normal bundle of* I.

Lemma 2.1. *Given a Dirac structure* $D \subset A \oplus A^*$ *and a subbundle* $I \subset A$ *which belongs to* D, *the existence of the bundle map* $\Omega^\#$ *restricted to* $I^\perp$ *is equivalent to a bivector field on the quotient bundle* A/I.

Therefore, we can define an equivalence relation on the space of characteristic pairs, by claiming that two characteristic pairs (I_1, Ω_1), (I_2, Ω_2) are equivalent if and only if

$$\begin{cases} I_1 = I_2 \equiv I \\ \Omega_1^\#(\alpha) - \Omega_2^\#(\alpha) \in I \qquad \forall \alpha \in I^\perp. \end{cases} \tag{4}$$

This definition leads then to the equivalence of the equivalence classes with the set of Dirac structures of a given Lie bialgebroid.

Theorem 2.1 ([12])**.** *Let* (A, A^*) *be a Lie bialgebroid and* D *a subbundle of* $A \oplus A^*$ *maximally isotropic with respect to the symmetric product (1), and corresponding to the characteristic pair class* (I, Ω). *Then,* D *is a Dirac structure if and only if*

- *I is a Lie subalgebroid.*
- *Ω satisfies the Maurer–Cartan type of equation:*

$$d_*\Omega + \tfrac{1}{2}[\Omega, \Omega] = 0 \mod I. \tag{5}$$

- *The following bracket is closed on* $\Gamma I^\perp$:

$$[\alpha, \beta] = [\alpha, \beta]_* + \mathcal{L}_{\Omega^\# \alpha}\beta - \mathcal{L}_{\Omega^\# \beta}\alpha - d(\Omega(\alpha, \beta)) \qquad \forall \alpha, \beta \in \Gamma I^\perp. \tag{6}$$

It is also easy to prove the following result about the different representatives of the class of characteristic pairs.

Proposition 2.2. *Consider a Lie bialgebroid* (A, A^*) *and a Dirac structure described by the equivalence class of characteristic pairs* $[(I, \Omega)]$. *Then, if one representative satisfies the conditions of theorem 2.1, so do all others.*

2.2. Courant algebroids

The double of any Lie bialgebroid $A \oplus A^*$ carries a geometric structure called Courant algebroid. There are two equivalent ways of defining such a structure:

- The original definition is given in [13]. It uses the symmetric structure of (1) and the bracket (2) introduced above and some compatibility relations among them.
- The definition given in [6, 21], which is based on the definition of a Leibniz algebra structure on the set of sections of the bundle $A \oplus A^*$. It requires fewer axioms than the previous one, and it does not use explicitly the operation (2).

Both definitions are equivalent and admit equivalent definitions of Dirac structures contained in the bundle $A \oplus A^*$, but the first one is, from our point of view, better adapted to the study of Dirac structures since it is the operation (2) that becomes the Lie algebra structure of the Dirac structure when restricted to the sections of the bundle D. This is why we proceed with the first definition.

Definition 2.4. *A* Courant algebroid *is a vector bundle $E \to M$, endowed with a non-degenerate symmetric bilinear form $(\cdot,\cdot)$, a skew-symmetric bracket $[\cdot,\cdot]$ and a mapping $\rho : E \to TM$ with the following properties:*

- *For any three sections $e_1, e_2, e_3 \in \Gamma E$, the obstruction to the Jacobi identity to hold can be computed as*
$$[e_1, [e_2, e_3]] + [e_3, [e_1, e_2]] + [e_2, [e_3, e_1]] = \mathfrak{d}T(e_1, e_2, e_3) \tag{7}$$
where
$$T(e_1, e_2, e_3) = \tfrac{1}{3}((e_1, [e_2, e_3]) + (e_3, [e_1, e_2]) + (e_2, [e_3, e_1])) \tag{8}$$
and $\mathfrak{d} : C^\infty(M) \to \Gamma E$ is the map defined by
$$\mathfrak{d} = \tfrac{1}{2}\beta^{-1}\rho^* d \tag{9}$$
with d the usual exterior differential of the base manifold M and β is the isomorphism $\beta : E \to E^*$ given by the bilinear form $(\cdot,\cdot)$.
- *For any two sections $e_1, e_2 \in \Gamma E$,*
$$\rho([e_1, e_2]) = [\rho(e_1), \rho(e_2)] \tag{10}$$
where the last bracket is the commutator of vector fields.
- *For any two sections $e_1, e_2 \in \Gamma E$ and any function $f \in C^\infty(M)$,*
$$[e_1, f e_2] = f[e_1, e_2] + (\rho(e_1)f)e_2 - (e_1, e_2)\mathfrak{d}f. \tag{11}$$
- *The operator $\mathfrak{d}$ is in the kernel of ρ, i.e. $\rho \circ \mathfrak{d} = 0$. This implies that, given any two functions $f, g \in C^\infty(M)$, $(\mathfrak{d}f, \mathfrak{d}g) = 0$.*
- *Given any three sections, $e, h_1, h_2 \in \Gamma E$,*
$$\rho(e)(h_1, h_2) = ([e, h_1] + \mathfrak{d}(e, h_1), h_2) + (h_1, [e, h_2] + \mathfrak{d}(e, h_2)). \tag{12}$$

Hence any Lie bialgebroid (A, A^*) yields a Courant algebroid structure on the Whitney sum $B = A \oplus A^*$ with the choices $\rho_B = \rho + \rho_*$, the natural symmetric bilinear form and the bracket (2) (see [13]). Therefore, the condition for the sections of the bundle D to become a Lie algebra is just

Definition 2.5. *Let the Courant algebroid $(E, \rho, (\cdot,\cdot), [\cdot,\cdot])$ be defined as above. Let $D \subset E$ be a maximally isotropic subbundle of E with respect to $(\cdot,\cdot)$. We say that D is a* Dirac structure *if it is closed for the product $[\cdot,\cdot]$.*

3. Deformation of Lie algebroids and Lie bialgebroids

In this section we present the main ideas required to define the deformation of Dirac structures in the last section, what is our main goal.

The meaning of the deformation. The geometrical meaning of the deformation can be followed in [5]. Consider a Lie algebra $(U, [\cdot,\cdot])$, and consider a linear deformation of the Lie bracket in the direction of a second Lie bracket $\omega : U \times U \to U$

$$[X, Y]_\lambda = [X, Y] + \lambda\omega(X, Y) \qquad X, Y \in U \quad \lambda \in \mathbb{R}.$$

If all the brackets $[\cdot,\cdot]_\lambda$ endow U with Lie algebra structures, we say that ω generates a deformation of the Lie algebra of U. We will assume now that to be the case.

We can also study a deformation of the space U itself, and study under what circumstances the deformed bracket is a Lie bracket homomorphic to the original one via a linear deformation such as

$$T_\lambda = \mathrm{Id} + \lambda N$$

where $N : U \to U$ is a linear operator.

Definition 3.1. *We shall call* trivial *those deformations which are homomorphic to the original one, i.e.* $T_\lambda[X, Y]_\lambda = [T_\lambda X, T_\lambda Y] \forall X, Y \in U$. *The triviality of the deformation is equivalent to asking that, for any* $X, Y \in U$*:*

$$[X, Y]_N \equiv \omega(X, Y) = [NX, Y] + [X, NY] - N[X, Y]$$
$$N[X, Y]_N \equiv N\omega(X, Y) = [NX, NY].$$

An operator $N : U \to U$ *which satisfies these conditions is called a* Nijenhuis operator. *The conditions are equivalent to ask the* Nijenhuis torsion $\mathcal{T}_N(X, Y) = [N(X), N(Y)] - N([X, N(Y)]) - N([N(X), Y]) + N^2([X, Y])$ *to vanish identically on any two elements* $X, Y \in U$.

Lemma 3.1 ([5])**.** *Consider a Nijenhuis operator N of the Lie algebra U. Then, a linear deformation of the Lie bracket of U in the direction of* $[X, Y]_N = [NX, Y] + [X, NY] - N[X, Y]$ *is trivial.*

3.1. Nijenhuis operators on Lie bialgebroids

3.1.1. Definition of Nijenhuis operators. The concept is well known [11, 7] as a simple generalization of the usual concept for vector fields. As we saw in the previous section, an operator is said to be of Nijenhuis type if its Nijenhuis torsion vanishes:

Definition 3.2. *Let* $(A, [\cdot,\cdot], \rho)$ *be a Lie algebroid. A linear transformation*

$$N : A \to A$$

is said to be a Nijenhuis transformation of A if and only if the torsion tensor $\mathcal{T}_N$ defined as

$$\mathcal{T}_N(X, Y) = [N(X), N(Y)] - N([X, N(Y)]) - N([N(X), Y]) + N^2([X, Y])$$
$$\forall X, Y \in \Gamma A \tag{13}$$

vanishes.

This is equivalent to having a new Lie structure on the sections of A, N being a homomorphism for them. But in the case of a Lie algebroid, this also implies that another Lie algebroid structure is available for A.

Lemma 3.2 ([7])**.** *Consider a Lie algebroid $(A, [\cdot,\cdot], \rho)$ and a Nijenhuis operator $N : A \to A$. Define the following bracket on the sections of A:*

$$[X, Y]_N = -N([X, Y]) + [X, N(Y)] + [N(X), Y]. \tag{14}$$

Consider the mapping

$$\hat{N} = \rho \circ N : A \to TM. \tag{15}$$

Then, $(A, [\cdot,\cdot]_N, \hat{N})$ is a Lie algebroid. This new Lie algebroid has a new exterior derivative, defined as

$$d^N = [i_N, d] = i_N \circ d - d \circ i_N \tag{16}$$

where i_N is the superderivation of degree zero on the forms $\Gamma\left(\bigwedge^{\bullet} A^*\right)$ defined as

$$i_N\theta(X_1, \ldots, X_p) = \sum_{i=1}^{p} \theta(X_1, \ldots, NX_i, X_{i+1}, \ldots, X_p). \tag{17}$$

The new structure also leads to a new Schouten bracket:

Proposition 3.1 ([11])**.** *Let N be a Nijenhuis operator on a Lie algebroid A. The Schouten bracket defined on $\Gamma\left(\bigwedge^{\bullet} A\right)$ by extension of the deformed bracket $[\cdot,\cdot]_N$ satisfies*

$$[Q, Q']_N = [i_{N^*}Q, Q'] + [Q, i_{N^*}Q'] - i_{N^*}[Q, Q'] \tag{18}$$

where i_{N^*} is the superderivation of degree zero on the sections of $\Gamma\left(\bigwedge^{\bullet} A\right)$ defined as

$$i_{N^*}\Psi(\theta_1, \ldots, \theta_p) = \sum_{i=1}^{p} \Psi(\theta_1, \ldots, N^*\theta_i, \theta_{i+1}, \ldots, \theta_p). \tag{19}$$

An important property for the next sections will be the possibility of combining deformations to define a new one. In [14], two Nijenhuis operators are said to be compatible if their product is also a Nijenhuis operator. That choice makes sense in their context since they are thinking in the definition of a chain of deformations obtained as a product.

We are interested in a different type of compatibility of deformations, regarding the sum of two Nijenhuis operators. The reasons for this choice shall become clear in the next sections. So, we define

Definition 3.3. *Let A be a Lie algebroid and N_1 and N_2 two Nijenhuis operators for it. We say that N_1 and N_2 are* sum-compatible *if the sum $N_1 + N_2$ is also a Nijenhuis operator for A.*

From the definition, it is trivial to prove the following result:

Lemma 3.3. *N_1 and N_2 are sum-compatible Nijenhuis operators if and only if*

$$N_2[X, Y]_{N_1} + N_1[X, Y]_{N_2} = [N_1X, N_2Y] + [N_2X, N_1Y] \qquad \forall X, Y \in A. \tag{20}$$

Proof. The definition of the deformed bracket is linear in the Nijenhuis operator, hence, the nontrivial condition is the homomorphic one. Writing down the expression associated with $N_1 + N_2$ and using the fact that N_1 and N_2 are Nijenhuis operators, the equivalence follows. □

It can immediately be seen that

Corollary 3.1. *Let A be a Lie algebroid. Then, any Nijenhuis operator N_1 is sum-compatible with the trivial deformation $N_2 = \lambda\mathrm{Id}$, where $\lambda \in \mathbb{R}$.*

Proof. It is trivial to see that (20) is equivalent to the condition in (14), which is satisfied because N_1 is of Nijenhuis type. □

3.2. How to define deformations of Lie bialgebroids

The next step is to consider the action of a Nijenhuis transformation on a Lie bialgebroid. Again, our objective is to apply these concepts to the deformation of Dirac structure in the following sections, therefore we consider now just those situations that may be interesting later.

In principle, we can consider two different transformations applied on each factor, i.e.

$$\begin{cases} N : A \to A \\ \Upsilon : A^* \to A^* \end{cases} \tag{21}$$

such that they define two new Lie algebroid structures on the factors, i.e. $(A, [\cdot,\cdot]_N, \hat{N})$ and $(A^*, [\cdot,\cdot]_\Upsilon, \hat{\Upsilon})$ are Lie algebroids. We must verify now whether they define a new Lie bialgebroid, i.e. the exterior derivatives are derivations for the deformed brackets. First of all, we must realize that the transformation on the Lie algebroid structures implies a transformation of the Lie algebroid cohomologies, i.e. there are two new exterior differentials d^N and d_*^Υ which are the ones to consider.

It is important to distinguish two different operators acting on each bundle: the Nijenhuis-like operators N, Υ and their duals:

$$(N)^* : A^* \to A^* \qquad (\Upsilon)^* : A \to A$$

which are defined as

$$\langle X, (N)^*\alpha\rangle = \langle N(X), \alpha\rangle \qquad \forall X \in \Gamma A \quad \forall \alpha \in \Gamma A^* \tag{22}$$

and

$$\langle (\Upsilon)^* X, \alpha\rangle = \langle X, \Upsilon\alpha\rangle \qquad \forall X \in \Gamma A \quad \forall \alpha \in \Gamma A^*. \tag{23}$$

Definition 3.4. *Consider a Lie bialgebroid (A, A^*) and a transformation of the type (21), such that N and Υ are trivial deformations of A and A^*. We say that the pair (N, Υ) defines a trivial deformation of the Lie bialgebroid structure if and only if the pair $((A, [\cdot,\cdot]_N, \hat{N}), (A^*, [\cdot,\cdot]_\Upsilon, \hat{\Upsilon}))$ defines a new Lie bialgebroid structure. We can write this fact by two equivalent conditions:*

$$\begin{cases} d_*^\Upsilon [X, Y]_N = [d_*^\Upsilon X, Y]_N + [X, d_*^\Upsilon Y]_N \\ d^N[\alpha, \beta]_\Upsilon = [d^N\alpha, \beta]_\Upsilon + [\alpha, d^N\beta]_\Upsilon. \end{cases} \tag{24}$$

It is important to realize that on the right-hand side of (24), the bracket involved is the deformed Schouten bracket of multisections, as defined in (18).

3.2.1. The simplest deformation.

Consider the simplest example of deformation of a Lie bialgebroid: a deformation of the Lie algebroid A which does not involve a deformation of the dual A^*. In such a case, the transformation is as follows:

$$\begin{cases} N \equiv N : A \to A \\ \Upsilon \equiv \mathrm{Id} : A^* \to A^*. \end{cases} \tag{25}$$

In these circumstances, the two new Lie structures are as follows:

$$\begin{cases} [X, Y]_N = -N[X, Y] + [NX, Y] + [X, NY] \qquad \forall X, Y \in \Gamma A \\ [\alpha, \beta]_\Upsilon = [\alpha, \beta]_* \qquad \forall \alpha, \beta \in \Gamma A^*. \end{cases} \tag{26}$$

And the corresponding exterior derivatives are

$$\begin{cases} d^N = i_N \circ d - d \circ i_N \\ d_*^\Upsilon = d_*. \end{cases} \tag{27}$$

The condition for (25) to define a deformation of the Lie bialgebroid (A, A^*) is then written in terms of the derivation of the deformed Lie structure on A or, equivalently, in terms of the deformed Lie structure of A^*:

$$\begin{cases} d_*[X, Y]_N = [d_*X, Y]_N + [X, d_*Y]_N & \forall X, Y \in \Gamma A \\ d^N[\alpha, \beta]_* = [d^N\alpha, \beta]_* + [\alpha, d^N\beta]_* & \forall \alpha, \beta \in \Gamma A^*. \end{cases} \tag{28}$$

As both conditions are equivalent, we are going to use just the first one:

By using the expression for the deformed bracket we can write

$$d_*[X, Y]_N = -d_*N[X, Y] + d_*[NX, Y] + d_*[X, NY].$$

For the last two terms, we can use the fact that d_* is a derivation for the original Lie bracket and write

$$d_*[NX, Y] = [d_*NX, Y] + [NX, d_*Y] \qquad d_*[X, NY] = [d_*X, NY] + [X, d_*NY].$$

We can write the action of N on the sections of A as

$$N(X) = i_{N^*}X \qquad \forall X \in \Gamma A.$$

In that case we obtain

$$d_*[X, Y]_N = -d_*i_{N^*}[X, Y] + [d_*i_{N^*}X, Y] + [i_{N^*}X, d_*Y] + [d_*X, i_{N^*}Y] + [X, d_*i_{N^*}Y].$$

The development of the right-hand side of the first equation of condition (28) by using the expression of the deformed Schouten bracket leads to

$$\begin{aligned}[d_*X, Y]_N + [X, d_*Y]_N = &-i_{N^*}[d_*X, Y] - i_{N^*}[X, d_*Y] \\ &+ [i_{N^*}d_*X, Y] + [d_*X, i_{N^*}Y] + [i_{N^*}X, d_*Y] + [X, i_{N^*}d_*Y].\end{aligned}$$

Comparing both sides, many terms cancel and we obtain

$$-d_*i_{N^*}[X, Y] + [X, d_*i_{N^*}Y] + [d_*i_{N^*}X, Y] = -i_{N^*}d_*[X, Y] + [X, i_{N^*}d_*Y] + [i_{N^*}d_*X, Y].$$

As the Schouten bracket is linear, we conclude that the first relation in (28) holds if and only if $[i_{N^*}, d_*]$ is a derivation for the original Lie bracket of A.

Therefore, we have proved

Theorem 3.1. *An operator* (N, Id) *defines a trivial deformation of the Lie bialgebroid* (A, A^*) *if and only if*

$$\delta_{NA} = i_{N^*} \circ d_* - d_* \circ i_{N^*} \tag{29}$$

is a derivation of the original Schouten bracket on A*. Equivalently, it can be said that* i_N *is a derivation of the* A^**-Schouten bracket. As* d *is assumed to also be a derivation for it, we conclude that the operator*

$$\delta_{NA^*} = i_N \circ d - d \circ i_N \tag{30}$$

is also a derivation of the A^**-Schouten bracket.*

3.2.2. General deformations. Finally, let us consider again a general deformation of type (21), i.e. the Nijenhuis operator for A and the Nijenhuis operator for A^* need not be duals of each other. From what we learned above, we can claim

Proposition 3.2. *A sufficient condition for* (N, Υ) *(where, of course, N is assumed to be a Nijenhuis operator for A and* Υ *a Nijenhuis operator for* A^**) to be a trivial deformation of the Lie bialgebroid* (A, A^*) *is that* i_Υ *is a derivation for* $[\,,\,]$ *and that* $d_*^\Upsilon N = N d_*^\Upsilon$*. Equivalently, for the dual objects we have that if* $d^N \Upsilon = \Upsilon d^N$ *and* i_N *is a derivation of* $[\cdot,\cdot]_*$*, then* (N, Υ) *will define a derivation for the Lie bialgebroid* (A, A^*).

Proof. Consider the definition of the differential $d_*^\Upsilon = [i_\Upsilon, d_*]$ and the Lie bracket $[\cdot,\cdot]_N$. As (A, A^*) is a Lie bialgebroid, d_* is a derivation for $[\cdot,\cdot]$. By hypothesis, so is i_Υ, which implies that d_*^Υ is also a derivation. Using the definition of the deformed bracket $[\cdot,\cdot]_N$ and the condition $d_*^\Upsilon N = N d_*^\Upsilon$, the conclusion follows. The relations for the dual objects are proved in the same way. □

As a corollary, now consider a deformation of the algebroid A defined by a Nijenhuis operator

$$N : A \to A$$

and the corresponding transformation defined on the dual bundle via the dual mapping, which is assumed to be also a Nijenhuis operator for A^*:

$$\Upsilon = N^* : A^* \to A^*.$$

Corollary 3.2. *Let* (A, A^*) *be a Lie bialgebroid. Consider the deformation*

$$\begin{cases} N : A \to A \\ N^* : A^* \to A^* \end{cases} \tag{31}$$

and assume that

- $d^N = i_N \circ d - d \circ i_N$ *is a derivation of the Lie bracket* $[\cdot,\cdot]_*$,
- $N^* d^N = d^N N^*$

or, equivalently,

- $d^{N^*} = i_{N^*} \circ d_* - d_* \circ i_{N^*}$ *is a derivation of the Lie bracket* $[\cdot,\cdot]$,
- $N d_*^{N^*} = d_*^{N^*} N$.

Then, the deformed structure $((A, N \circ \rho, [\cdot,\cdot]_N), (A^*, N^* \circ \rho_*, [\cdot,\cdot]_{N^*})$*, is a Lie bialgebroid.*

3.3. Application: Poisson–Nijenhuis structures

As an example of deformation of a Lie bialgebroid, let us review Kosmann–Schwarzbach's construction in [10]. The goal is to test our results on a well-known situation.

Consider a Poisson manifold (M, Λ). It is well known that the cotangent bundle T^*M can be endowed with a Lie algebroid structure, by using $\Lambda^\# : T^*M \to TM$ as an anchor mapping and the Lie product:

$$[\alpha, \beta]_\Lambda = \mathcal{L}_{\Lambda^\#(\alpha)}\beta - \mathcal{L}_{\Lambda^\#(\beta)}\alpha - d(\Lambda(\alpha, \beta)) \qquad \forall \alpha, \beta \in \Gamma T^*M. \tag{32}$$

The tangent bundle TM can be endowed trivially with a structure of Lie algebroid, and hence the pair (TM, T^*M) is a pair of dual Lie algebroids. It is also possible to see that the corresponding exterior derivatives are compatible, in such a way that the pair (TM, T^*M) endowed with the two structures above is a Lie bialgebroid.

We can now consider a deformation of the canonical structure of Lie algebroid on TM by means of a Nijenhuis operator. The effect on the Poisson bivector Λ suggests the introduction of a special type of deformation, the so-called compatible deformations, which yield the concept of Poisson–Nijenhuis manifolds (see [11, 16]):

Definition 3.5. *A* Poisson–Nijenhuis *manifold (or PN manifold) is a Poisson manifold (M, Λ) and a Nijenhuis operator $N : TM \to TM$ which is* compatible *with the Poisson structure, compatible meaning:*

- $N\Lambda^\# = \Lambda^\# \circ N^*$, *what implies that the tensor $N\Lambda$ is skew-symmetric.*
- *Magri's concomitant (see [11, 16]) vanishes, i.e.* $C(\Lambda, N)(\alpha, \beta) \equiv [\alpha, \beta]_{N\Lambda} - ([N^*\alpha, \beta]_\Lambda + [\alpha, N^*\beta]_\Lambda - N^*[\alpha, \beta]_\Lambda) = 0$.

The advantage of PN manifolds is the behaviour of the Lie bialgebroid structures on them.

Theorem 3.2 ([10])**.** *Consider a Poisson manifold (M, Λ) and a Nijenhuis operator N. The pair (TM, T^*M) endowed with the structures*

$$\begin{cases} TM, \rho = N, [X, Y]_N = [NX, Y] + [X, NY] - N[X, Y] & \forall X, Y \in \Gamma TM \\ T^*M, \rho = \Lambda^\#, [\alpha, \beta]_\Lambda = \mathcal{L}_{\Lambda^\#(\alpha)}\beta - \mathcal{L}_{\Lambda^\#(\beta)}\alpha - d(\Lambda(\alpha, \beta)) & \forall \alpha, \beta \in \Gamma T^*M \end{cases} \tag{33}$$

is a Lie bialgebroid if and only if M is a PN manifold.

It is simple to verify that the conditions described above are satisfied in this deformation of Lie bialgebroids. The conditions are the same contained in proposition 3.2 of [10], but presented in a slightly different way. Roughly speaking, the conditions presented there correspond to the definition of a deformation of a Lie bialgebroid (i.e. the transformed exterior differentials are derivations of the transformed Schouten brackets).

3.4. Sum-compatible deformations

We can also consider the concept of sum-compatible deformations in this case.

Definition 3.6. *Let (A, A^*) be a Lie bialgebroid and (N_1, Υ_1) and (N_2, Υ_2) two pairs of Nijenhuis operators for it which define a trivial deformation of the Lie bialgebroid structure. We say that they are* sum-compatible *if the sum $(N_1 + N_2, \Upsilon_1 + \Upsilon_2)$ defines also a trivial deformation of (A, A^*), i.e.*

- *N_1 and N_2 are sum-compatible Nijenhuis operators for A,*
- *Υ_1 and Υ_2 are sum-compatible Nijenhuis operators for A^*,*
- *for any $X, Y \in A$,*

$$d_*^{\Upsilon_1}[X, Y]_{N_2} + d_*^{\Upsilon_2}[X, Y]_{N_1} = [d_*^{\Upsilon_1}X, Y]_{N_2} + [X, d_*^{\Upsilon_1}Y]_{N_2} + [d_*^{\Upsilon_2}X, Y]_{N_1} + [X, d_*^{\Upsilon_2}Y]_{N_1}. \tag{34}$$

From that expression, it is trivial to see

Lemma 3.4. *Let (A, A^*) be a Lie bialgebroid and let $\mathcal{N}_1 = (N_1, \Upsilon_1)$ and $\mathcal{N}_2 = (N_2, \Upsilon_2)$ define trivial deformations. If $(\lambda \mathrm{Id}, \Upsilon_2)$ and $(N_2, \lambda \mathrm{Id})$, where $\lambda \in \mathbb{R}$, are trivial deformations of $((A, [\cdot,\cdot]_{N_1}, \hat{N}_1), (A^*, [\cdot,\cdot]_{\Upsilon_1}, \hat{\Upsilon}_1))$, and $(\lambda\,\mathrm{Id}, \Upsilon_1)$ and $(N_1, \lambda\,\mathrm{Id})$, where $\lambda \in \mathbb{R}$, are trivial deformations of $((A, [\cdot,\cdot]_{N_2}, \hat{N}_2), (A^*, [\cdot,\cdot]_{\Upsilon_2}, \hat{\Upsilon}_2))$ then $\mathcal{N}_1$ and $\mathcal{N}_2$ are sum-compatible deformations.*

Analogously to the equivalence of deformations in Lie algebroids, we obtain a simple result:

Lemma 3.5. *Let (A, A^*) be a Lie bialgebroid and let (N, Υ) define a trivial deformation. Then, $(N, \lambda\,\mathrm{Id}_{\Gamma A^*})$ and $(\lambda\,\mathrm{Id}_{\Gamma A}, \Upsilon)$ define sum-compatible deformations, where $\lambda \in \mathbb{R}$.*

Proof. From corollary 3.1, we know that N and Υ are sum-compatible with the corresponding identity. Then, the only point to prove is that (34) is satisfied. But it reads now

$$\lambda d_*[X, Y] + d_*^{\Upsilon}[X, Y]_N = \lambda[d_* X, Y] + \lambda[X, d_* Y] + [d_*^{\Upsilon} X, Y]_N + [X, d_*^{\Upsilon} Y]_N \qquad \forall X, Y \in \Gamma A.$$

And this relation is satisfied because as (A, A^*) is a Lie bialgebroid for both the deformed and the undeformed structures, we have

$$d_*[X, Y] = [d_* X, Y] + [X, d_* Y] \qquad \forall X, Y \in \Gamma A$$

and

$$d_*^{\Upsilon}[X, Y]_N = [d_*^{\Upsilon} X, Y]_N + [X, d_*^{\Upsilon} Y]_N \qquad \forall X, Y \in \Gamma A.$$

□

4. Deformation of the double of a Lie bialgebroid and Courant algebroids

There is an important difference that explains the different philosophies which oppose the present paper and [1]: as the main aim of [1] is to study the deformation of Courant algebroids, they use as a basic tool the deformation of Leibniz algebras. With that setting, the authors study the implications that this deformation has on the Dirac structures defined on the Courant algebroid which is being deformed, and they give a general definition of Dirac–Nijenhuis manifolds. Then, they proceed to study two very simple examples of deformation of Dirac structures. On the other hand, our aim is somehow the opposite. We are interested in the deformation of Dirac structures as Lie algebroids, and therefore we need to deform the operation (2). This can also have implications on the Courant algebroid structure of the double $A \oplus A^*$, and hence we also study some of its deformations. Finally, we reach a definition of Dirac–Nijenhuis structures which is equivalent to the one in [1], but from our point of view our method is more flexible in this context and is applied to more general examples. This is why we chose the original definition of the Courant algebroid.

Consider then the double of the Lie bialgebroid (A, A^*), i.e. the bundle $B = A \oplus A^*$. This bundle is not endowed with a Lie algebroid structure, but with a Courant algebroid one. As Dirac structures are defined as suitable subbundles of B, we need to consider deformations of B in order to deform these subbundles. Again, we do not consider the most general deformations of B, but some particular types which are of interest in the next section to deform Dirac structures. Therefore, this scheme may be less general than the approach followed in [1] from the Courant algebroid point of view, but it proves to be useful in the study of the deformation of quite general Dirac structures.

In the description of the Courant algebroid structure of $A \oplus A^*$ above, let us consider the skew-symmetric bracket (2). Even if it does not define a Lie algebra structure it is still a skewsymmetric one, and we can still consider a deformation of it, and ask it to be trivial in the sense of being homomorphic to the original one.

Definition 4.1. *Consider the double of a Lie bialgebroid $B = A \oplus A^*$ and the bracket (2). Consider a deformation of the structure in the form*

$$[b_1, b_2]_\lambda = [b_1, b_2] + \lambda[b_1, b_2]_{\mathcal{N}} \qquad \forall b_1, b_2 \in \Gamma B$$

where $[\cdot,\cdot]_{\mathcal{N}} = [\mathcal{N}\cdot,\cdot] + [\cdot,\mathcal{N}\cdot] - \mathcal{N}[\cdot,\cdot]$ *and* $\mathcal{N} : \Gamma B \to \Gamma B$ *is a linear operator. Consider also a deformation of the set of sections of B in the form:*

$$T_\lambda = \mathrm{Id}_{\Gamma B} + \lambda\mathcal{N}.$$

Then, we say that the deformation is trivial if

$$T_\lambda[b_1, b_2]_\lambda = [T_\lambda b_1, T_\lambda b_2] \quad \forall b_1, b_2 \in \Gamma B.$$

We can also consider a certain Nijenhuis-like property for the operator $\mathcal{N}$. Formally, we can still define the Nijenhuis torsion in the way we did above:

$$\mathcal{T}_{\mathcal{N}}(X, Y) = [\mathcal{N}(X), \mathcal{N}(Y)] - \mathcal{N}([X, \mathcal{N}(Y)]) - \mathcal{N}([\mathcal{N}(X), Y]) + \mathcal{N}^2([X, Y]) \qquad \forall X, Y \in \Gamma B. \tag{35}$$

To ask this quantity to vanish identically, is equivalent to asking the operator $\mathcal{N}$ to satisfy:

$$\mathcal{N}[X, Y]_{\mathcal{N}} = [\mathcal{N}X, \mathcal{N}Y]. \tag{36}$$

In the following, we will also call the Nijenhuis operator an operator $\mathcal{N} : \Gamma B \to \Gamma B$ which satisfies this condition. Of course, it is clear that many of the properties of the pure Nijenhuis operator will not be shared by these, but the formal definition is still possible. It is still true, though, that these operators define trivial deformations of the skewsymmetric structure of B, as the usual Nijenhuis operators define trivial deformations of Lie structures.

If we consider a deformation of the Lie bialgebroid (A, A^*) in the sense discussed above, we know that such a deformation provides new Lie structures on ΓA and ΓA^*, new Lie derivatives $\mathcal{L}^{\Upsilon}_{*\,\alpha}$, $\mathcal{L}^N_X$ and exterior differentials d^N and d^{Υ}_*. This implies that we can consider the analogue of the bracket (2), for the new Lie bialgebroid structure, i.e.

$$[(X_1, \alpha_1), (X_2, \alpha_2)]^{\mathcal{N}}_{\diamond} = ([X_1, X_2]_N + [(X_1, \alpha_1), (X_2, \alpha_2)]_{\mathcal{L}^{\Upsilon}_*}, [(X_1, \alpha_1), (X_2, \alpha_2)]_{\mathcal{L}^N} + [\alpha_1, \alpha_2]_{\Upsilon}) \tag{37}$$

with

$$[(X_1, \alpha_1), (X_2, \alpha_2)]_{\mathcal{L}^{\Upsilon}_*} = \mathcal{L}^{\Upsilon}_{*\,\alpha_1} X_2 - \mathcal{L}^{\Upsilon}_{*\,\alpha_2} X_1 - \tfrac{1}{2} d^{\Upsilon}_*\big(i_{X_2}\alpha_1 - i_{X_1}\alpha_2\big) \tag{38}$$

and

$$[(X_1, \alpha_1), (X_2, \alpha_2)]_{\mathcal{L}^N} = \mathcal{L}^N{}_{X_1}\alpha_2 - \mathcal{L}^N{}_{X_2}\alpha_1 + \tfrac{1}{2} d^N\big(i_{X_2}\alpha_1 - i_{X_1}\alpha_2\big) \tag{39}$$

and where the brackets $[\cdot,\cdot]_N$ and $[\cdot,\cdot]_{\Upsilon}$ are the brackets of the Lie algebroids $(A, [\cdot,\cdot]_N, \hat{N})$ and $(A^*, [\cdot,\cdot]_{\Upsilon}, \hat{\Upsilon})$ defined in (14).

But we can also try to see the above deformation as a deformation of the original Courant algebroid structure of $B = A \oplus A^*$, in the sense of definition 4.1 using the bracket above as the linear term of the deformation. The condition for the deformation $(T_\lambda, [\cdot,\cdot]_\lambda)$ to be trivial will be that the operator $\mathcal{N}$ in T_λ is a Nijenhuis operator for the original bracket. This implies

- $[(X_1, \alpha_1), (X_2, \alpha_2)]^{\mathcal{N}}_{\diamond} = [\mathcal{N}(X_1, \alpha_1), (X_2, \alpha_2)]_{\diamond} + [(X_1, \alpha_1), \mathcal{N}(X_2, \alpha_2)]_{\diamond} - \mathcal{N}[(X_1, \alpha_1), (X_2, \alpha_2)]_{\diamond}$.
- $\mathcal{N}[(X_1, \alpha_1), (X_2, \alpha_2)]^{\mathcal{N}}_{\diamond} = [\mathcal{N}(X_1, \alpha_1), \mathcal{N}(X_2, \alpha_2)]_{\diamond}$.

But it turns out that not all deformations of the Lie algebroids A and A^* also define a deformation for the double. Some conditions arise from the double structure:

Theorem 4.1. *Consider the double of a Lie bialgebroid* $A \oplus A^*$ *and consider two Nijenhuis operators N for A and* Υ *for* A^**. Then, if*

- $N^* + \Upsilon = 2\lambda_1$*, where* $\lambda_1 \in \mathbb{R}$*,*
- $N^2 = \lambda_2$*, where* $\lambda_2 \in \mathbb{R}$*,*

then $\mathcal{N} = N \times \Upsilon : A \oplus A^* \to A \oplus A^*$ *is a Nijenhuis operator for* $A \oplus A^*$*.*

Proof. As N and Υ are supposed to be Nijenhuis operators for A and A^*, respectively, the first and fourth terms on the right-hand side of (37) correspond trivially to a torsion-free deformation. We have to consider only the second and third terms. As both are formally analogous, we are going to study only the combination $[(NX_1, \Upsilon\alpha_1), (X_2, \alpha_2)]_{\mathcal{L}} + [(X_1, \alpha_1), (NX_2, \Upsilon\alpha_2)]_{\mathcal{L}} - \Upsilon[(X_1, \alpha_1), (X_2, \alpha_2)]_{\mathcal{L}}$.

If we now compute each term separately when acting on some element of A, we have

$$\begin{aligned}\langle Y, [(NX_1, \Upsilon\alpha_1), (X_2, \alpha_2)]_{\mathcal{L}}\rangle &= \left\langle Y, \mathcal{L}_{NX_1}\alpha_2 - \mathcal{L}_{X_2}\Upsilon\alpha_1 + \tfrac{1}{2}d\left(i_{X_2}\Upsilon\alpha_1 - i_{NX_1}\alpha_2\right)\right\rangle\\ &= \rho(NX_1)\alpha_2(Y) - \alpha_2([NX_1, Y]) - \rho(X_2)\alpha_1(\Upsilon^*Y) + \alpha_1(\Upsilon^*[X_2, Y])\\ &\quad + \tfrac{1}{2}\rho(Y)\alpha_1(\Upsilon^*X_2) - \tfrac{1}{2}\rho(Y)\alpha_2(NX_1)\end{aligned}$$

$$\begin{aligned}\langle Y, [(X_1, \alpha_1), (NX_2, \Upsilon\alpha_2)]_{\mathcal{L}}\rangle &= \left\langle Y, \mathcal{L}_{X_1}\Upsilon\alpha_2 - \mathcal{L}_{NX_2}\alpha_1 + \tfrac{1}{2}d\left(i_{NX_2}\alpha_1 - i_{X_1}\Upsilon\alpha_2\right)\right\rangle\\ &= \rho(X_1)\alpha_2(\Upsilon^*Y) - \alpha_2(\Upsilon^*[X_1, Y]) - \rho(NX_2)\alpha_1(Y) + \alpha_1([NX_2, Y])\\ &\quad + \tfrac{1}{2}\rho(Y)\alpha_1(NX_2) - \tfrac{1}{2}\rho(Y)\alpha_2(\Upsilon^*X_1)\end{aligned}$$

$$\begin{aligned}\langle Y, \Upsilon[(X_1, \alpha_1), (X_2, \alpha_2)]_{\mathcal{L}}\rangle &= \rho(X_1)\alpha_2(\Upsilon^*Y) - \alpha_2([X_1, \Upsilon^*Y]) - \rho(X_2)\alpha_1(\Upsilon^*Y)\\ &\quad + \alpha_1([X_2, \Upsilon^*Y]) + \tfrac{1}{2}\rho(\Upsilon^*Y)\alpha_1(X_2) - \tfrac{1}{2}\rho(\Upsilon^*Y)\alpha_2(X_1).\end{aligned}$$

Without loss of generality, we can write condition $N^* + \Upsilon = 2\lambda_1$ as $\mathcal{N} = (\lambda_1 + N) \times (\lambda_1 - N^*)$. This implies

$$\begin{aligned}&\langle Y, [((\lambda_1 + N)X_1, (\lambda_1 - N^*)\alpha_1), (X_2, \alpha_2)]_{\mathcal{L}} + [(X_1, \alpha_1), ((\lambda_1 + N)X_2, (\lambda_1 - N^*)\alpha_2)]_{\mathcal{L}}\\ &\qquad - (\lambda_1 - N^*)[(X_1, \alpha_1), (X_2, \alpha_2)]_{\mathcal{L}}\rangle\\ &\quad = \rho((\lambda_1 + N)X_1)\alpha_2(Y) - \alpha_2([(\lambda_1 + N)X_1, Y]) - \alpha_2((\lambda_1 - N)[X_1, Y])\\ &\qquad + \alpha_2([X_1, (\lambda_1 - N)Y]) + \alpha_1((\lambda_1 - N)[X_2, Y]) - \rho((\lambda_1 + N)X_2)\alpha_1(Y)\\ &\qquad + \alpha_1([(\lambda_1 + N)X_2, Y]) - \alpha_1([X_2, (\lambda_1 - N)Y]) + \tfrac{1}{2}\rho(Y)\alpha_1((\lambda_1 - N)X_2)\\ &\qquad - \tfrac{1}{2}\rho(Y)\alpha_2((\lambda_1 + N)X_1) + \tfrac{1}{2}\rho(Y)\alpha_1((\lambda_1 + N)X_2) - \tfrac{1}{2}\rho(Y)\alpha_2((\lambda_1 - N)X_1)\\ &\qquad - \tfrac{1}{2}\rho((\lambda_1 - N)Y)\alpha_1(X_2) + \tfrac{1}{2}\rho((\lambda_1 - N)Y)\alpha_2(X_1).\end{aligned}$$

Now we use

$$\begin{aligned}&-\alpha_2([(\lambda_1 + N)X_1, Y]) - \alpha_2((\lambda_1 - N)[X_1, Y]) + \alpha_2([X_1, (\lambda_1 - N)Y])\\ &\quad = -\alpha_2([(\lambda_1 + N)X_1, Y] + [X_1, (N - \lambda_1)Y] - (N - \lambda_1)[X_1, Y])\\ &\quad = -\alpha_2([(\lambda_1 + N)X_1, Y] + [X_1, (N + \lambda_1)Y] - (N + \lambda_1)[X_1, Y])\end{aligned}$$

and the analogous relation for α_1 to write finally that

$$\begin{aligned}&\langle Y, [((\lambda_1 + N)X_1, (\lambda_1 - N^*)\alpha_1), (X_2, \alpha_2)]_{\mathcal{L}} + [(X_1, \alpha_1), ((\lambda_1 + N)X_2, (\lambda_1 - N^*)\alpha_2)]_{\mathcal{L}}\\ &\qquad - (\lambda_1 - N^*)[(X_1, \alpha_1), (X_2, \alpha_2)]_{\mathcal{L}}\rangle\\ &\quad = \rho((\lambda_1 + N)X_1)\alpha_2(Y) - \alpha_2\left([X_1, Y]_{\lambda_1+N}\right) - \rho((\lambda_1 + N)X_2)\alpha_1(Y)\\ &\qquad + \alpha_1\left([X_2, Y]_{\lambda_1+N}\right) + \tfrac{1}{2}\rho((\lambda_1 + N)Y)\alpha_1(X_2) - \tfrac{1}{2}\rho((\lambda_1+N)Y)\alpha_2(X_1)\\ &\quad = \left\langle Y, \mathcal{L}^{\lambda_1+N}_{X_1}\alpha_2 - \mathcal{L}^{\lambda_1+N}_{X_2}\alpha_1 + \tfrac{1}{2}d^{\lambda_1+N}(\alpha_1(X_2) - \alpha_2(X_1))\right\rangle.\end{aligned}$$

It is also possible to check whether the Nijenhuis torsion vanishes

$$\begin{aligned}&\left\langle Y, (\lambda_1 - N^*)[(X_1, \alpha_1), (X_2, \alpha_2)]_{\mathcal{L}_{\lambda_1+N}}\right\rangle = \left\langle(\lambda_1 - N)Y, [(X_1, \alpha_1), (X_2, \alpha_2)]_{\mathcal{L}_{\lambda_1+N}}\right\rangle\\ &\quad = \rho((\lambda_1 + N)X_1)((\lambda_1 - N^*)\alpha_2)(Y) - \rho((\lambda_1 + N)X_2)((\lambda_1 - N^*)\alpha_1)(Y)\\ &\qquad - \alpha_2([(\lambda_1 + N)X_1, (\lambda_1 - N)Y] + \left[X_1, \left(\lambda_1^2 - N^2\right)Y\right]\\ &\qquad - (\lambda_1 + N)[X_1, (\lambda_1 - N)Y]) + \alpha_1([(\lambda_1 + N)X_2, (\lambda_1 - N)Y]\end{aligned}$$

$$
\begin{aligned}
&+\left[X_2, \left(\lambda_1^2 - N^2\right)Y\right] - (\lambda_1 + N)[X_2, (\lambda_1 - N)Y]) \\
&+\tfrac{1}{2}\rho\left(\left(\lambda_1^2 - N^2\right)Y\right)\alpha_1(X_2) - \tfrac{1}{2}\rho\left(\left(\lambda_1^2 - N^2\right)Y\right)\alpha_2(X_1).
\end{aligned}
\tag{40}
$$

Meanwhile, we also have

$$
\begin{aligned}
&\langle Y, [((\lambda_1 + N)X_1, (\lambda_1 - N^*)\alpha_1), ((\lambda_1 + N)X_2, (\lambda_1 - N^*)\alpha_2)]_{\mathcal{L}}\rangle \\
&\quad = \rho((\lambda_1 + N)X_1)((\lambda_1 - N^*)\alpha_2)(Y) - \rho((\lambda_1 + N)X_2)((\lambda_1 - N^*)\alpha_1)(Y) \\
&\qquad - \alpha_2((\lambda_1 - N)[(\lambda_1 + N)X_1, Y]) + \alpha_1((\lambda_1 - N)[(\lambda_1 + N)X_2, Y]) \\
&\qquad + \tfrac{1}{2}\rho(Y)(\lambda_1 - N^*)\alpha_1((\lambda_1 + N)X_2) - \tfrac{1}{2}\rho(Y)(\lambda_1 - N^*)\alpha_2((\lambda_1 + N)X_1) \\
&\quad = \rho((\lambda_1 + N)X_1)((\lambda_1 - N^*)\alpha_2)(Y) - \rho((\lambda_1 + N)X_2)((\lambda_1 - N^*)\alpha_1)(Y) \\
&\qquad - \alpha_2((\lambda_1 - N)[(\lambda_1 + N)X_1, Y]) + \alpha_1((\lambda_1 - N)[(\lambda_1 + N)X_2, Y]) \\
&\qquad + \tfrac{1}{2}\rho(Y)\left(\lambda_1^2 - N^2\right)\alpha_1(X_2) - \tfrac{1}{2}\rho(Y)\left(\lambda_1^2 - N^2\right)\alpha_2(X_1).
\end{aligned}
\tag{41}
$$

Hence comparing (40) and (41), we see that the condition of homomorphism is fulfilled if

$$
\begin{aligned}
&[(\lambda_1 + N)X_1, (\lambda_1 - N)Y] + \left[X_1, \left(\lambda_1^2 - N^2\right)Y\right] - (\lambda_1 + N)[X_1, (\lambda_1 - N)Y]) \\
&\quad = (\lambda_1 - N)[(\lambda_1 + N)X_1, Y])
\end{aligned}
$$

and analogously for X_2. But we have, for instance for the X_1 case (the other is analogous), and taking into account that N is a Nijenhuis operator for A:

$$
\begin{aligned}
&[(\lambda_1 + N)X_1, (\lambda_1 - N)Y] + \left[X_1, \left(\lambda_1^2 - N^2\right)Y\right] - (\lambda_1 + N)[X_1, (\lambda_1 - N)Y]) \\
&\quad = \lambda_1^2[X_1, Y] + \lambda_1([NX_1, Y] - [X_1, NY]) - [NX_1, NY] + \left(\lambda_1^2 - \lambda_2\right)[X_1, Y] \\
&\qquad - \lambda_1^2[X_1, Y] - \lambda_1(N[X_1, Y] - [X_1, NY]) + N[X_1, NY] \\
&\quad = \lambda_1([NX_1, Y] - N[X_1, Y]) + \lambda_1^2[X_1, Y] - N^2[X_1, Y] - [NX_1, NY] \\
&\qquad + N[X_1, NY] = \lambda_1([NX_1, Y] - N[X_1, Y]) + \lambda_1^2[X_1, Y] - N[NX_1, Y] \\
&\quad = (\lambda_1 - N)[(\lambda_1 + N)X_1, Y]).
\end{aligned}
$$

And this implies that the Nijenhuis torsion vanishes. This proves the result. □

Remark. We have mentioned already that the approach proposed in [1] is more suitable to study the deformation of Courant algebroids than ours. Therefore, it is not surprising that this result can be obtained easily within that framework.

Finally, we can also consider the concept of sum-compatible deformations in this case.

Definition 4.2. *Let (A, A^*) be a Lie bialgebroid and $\mathcal{N}_1$ and $\mathcal{N}_2$ two Nijenhuis operators for the skew-symmetric bracket (37) in the double $A \oplus A^*$. We say that they are* sum-compatible *if the sum $\mathcal{N}_1 + \mathcal{N}_2$ also defines a trivial deformation of (37).*

5. Deformation of Dirac structures

The deformation of the double of a Lie algebroid as seen above leads naturally to the consideration of the deformation of Dirac structures. Being Courant algebroids, both the undeformed and the deformed doubles will define Dirac structures for the subset of sections for which the skew-symmetric operation is closed. Naturally, in both cases we will have Lie algebroid structures. But of course, not all trivial deformations of the double of the Lie bialgebroid lead to trivial deformations of D. Conversely, we can also consider deformations of the double of the Lie bialgebroid, which not being trivial for the whole set, are trivial for the sections of the Dirac structure. We are presenting these results divided into two sections: in this one, we will consider the case of a trivial deformation of the double of a Lie bialgebroid, and study under what circumstances that deformation defines a trivial deformation of the Dirac structure. In the next section, we will consider a deformation of the Dirac structure per se, without considering the effect that such a deformation has on the double.

5.1. Preliminaries

Therefore, our goal now is to consider how the deformation of the double of a Lie bialgebroid affects a Dirac structure $D \subset A \oplus A^*$ defined by a characteristic pair (I, Ω) as $D = I \oplus \mathrm{graph}(\Omega)|_{I^\perp}$. Evidently, the condition to ask is the new skew-symmetric bracket (37) of the double to be closed for the sections of D. It is important to remark that closeness ensures that the deformed bracket will be of Lie type, since the new structure will be defined with respect to the deformed Courant algebroid structure of the double.

From the previous section, we know that, given a Lie bialgebroid (A, A^*) and a deformation defined by the pair (N, Υ), the operator $\mathcal{N} = N \times \Upsilon : A \oplus A^* \to A \oplus A^*$ defines a trivial deformation of the double of the Lie bialgebroid if $\mathcal{N}$ is a Nijenhuis operator with respect to the skew-symmetric bracket (2). We also saw that a sufficient condition for that is

- $N^* + \Upsilon \in \mathbb{R}$.
- $N^2 \in \mathbb{R}$.

Remember that the skewsymmetric operation (2) is not a Lie structure on $A \oplus A^*$. Nonetheless, it is indeed a Lie structure when restricted to the sections of a Dirac structure. Hence, those deformations of the double which define an inner operation on the sections of D will define a trivial deformation of the Lie algebroid structure D is endowed with.

In the following, we are going to use the description of D in terms of characteristic pairs. Hence, we denote by (I, Ω) a characteristic pair which represents D. It is important to remember now that the characteristic pair determines the actual subbundle of $A \oplus A^*$ the Dirac structure is defined on. Hence, if we are interested in the definition of a deformation of a given Dirac structure, we will need that the sections of D are stable under the action of the transformation $\mathcal{N}$. This implies

Lemma 5.1. *Consider a deformation $\mathcal{N} = N \times \Upsilon$ of the double of a Lie bialgebroid (A, A^*) and a Dirac structure $D \subset A \oplus A^*$ with a characteristic pair (I, Ω). Then, $\mathcal{N}$ preserves the subbundle $I \oplus \mathrm{graph}(\Omega)|_{I^\perp}$ if and only if*

$$N\Omega^\# = \Omega^\# \circ \Upsilon \qquad N(I) \subset I \qquad \Upsilon(I^\perp) \subset I^\perp. \tag{42}$$

Proof. The action of $\mathcal{N}$ on D is of the form:

$$\mathcal{N}(X, \alpha) = (NX, \Upsilon\alpha).$$

Restricting the action to the graph of the bivector and assuming the stability conditions, the conclusion follows. □

Let us consider then the expression of the deformed skew-symmetric operation on the subbundle of $A \oplus A^*$ defined by the characteristic pair (I, Ω):

$$\begin{aligned} &[(X_\alpha + \Omega^\#(\alpha), \alpha), (X_\beta + \Omega^\#(\beta), \beta)]_\diamond^{\mathcal{N}} \\ &\quad = \big([X_\alpha + \Omega^\#(\alpha), X_\beta + \Omega^\#(\beta)]_N + [(X_\alpha + \Omega^\#(\alpha), \alpha), (X_\beta + \Omega^\#(\beta), \beta)]_{\mathcal{L}_{\mathcal{D}_*}^{\Upsilon}}, \\ &\qquad [(X_\alpha + \Omega^\#(\alpha), \alpha), (X_\beta + \Omega^\#(\beta), \beta)]_{\mathcal{L}_{\mathcal{D}}^{N}} + [\alpha, \beta]_\Upsilon\big) \end{aligned} \tag{43}$$

where now, the diagonal factors are

$$\begin{aligned}[(X_1,\alpha_1),(X_2,\alpha_2)]_{\mathcal{L}_{\mathcal{D}^N}} &= [\mathcal{N}(X_1,\alpha_1),(X_2,\alpha_2)]_{\mathcal{L}} \\ &\quad + [(X_1,\alpha_1),\mathcal{N}(X_2,\alpha_2)]_{\mathcal{L}} - \mathcal{N}[(X_1,\alpha_1),(X_2,\alpha_2)]_{\mathcal{L}} \end{aligned} \tag{44}$$

and

$$\begin{aligned}[(X_1,\alpha_1),(X_2,\alpha_2)]_{\mathcal{L}_{\mathcal{D}_*^\Upsilon}} &= [\mathcal{N}(X_1,\alpha_1),(X_2,\alpha_2)]_{\mathcal{L}^*} \\ &\quad + [(X_1,\alpha_1),\mathcal{N}(X_2,\alpha_2)]_{\mathcal{L}^*} - \mathcal{N}[(X_1,\alpha_1),(X_2,\alpha_2)]_{\mathcal{L}^*} \end{aligned} \tag{45}$$

and where the brackets $[\cdot,\cdot]_N$ and $[\cdot,\cdot]_\Upsilon$ are the brackets of the Lie algebroids $(A,[\cdot,\cdot]_N,\hat{N})$ and $(A^*,[\cdot,\cdot]_\Upsilon,\hat{\Upsilon})$ defined in (14).

As a conclusion, we just want to force the operation to be closed in D and define a trivial deformation of its Lie structure. This implies

- The operation must be closed in $D = I \oplus \mathrm{graph}(\Omega)|_{I^\perp}$.
- The Nijenhuis torsion vanishes on D,

$$\mathcal{N}[(X_\alpha+\Omega^\#(\alpha),\alpha),(X_\beta+\Omega^\#(\beta),\beta)]_\diamond^{\mathcal{N}} = [\mathcal{N}(X_\alpha+\Omega^\#(\alpha),\alpha),\mathcal{N}(X_\beta+\Omega^\#(\beta),\beta)]_\diamond. \tag{46}$$

But this condition will be granted for those $\mathcal{N}$ which define a trivial deformation of the double.

Therefore, for those deformations which are trivial for the double, it is enough to check for the closeness condition.

5.2. The closeness condition

Let us now consider one by one the conditions, and using the linearity of the bracket, we will differentiate each type of bracket (two elements of the graph of Ω, one element of the graph and one element of I and two elements of I). In each case, we will study whether the deformations which are trivial for the Dirac structure are also trivial for the double of the Lie bialgebroid. For all the computations, we need the following result:

Lemma 5.2. *Consider a Dirac structure $D = I \oplus \mathrm{graph}(\Omega)|_{I^\perp}$ as above. We can write the term $[(\Omega^\#(\alpha),\alpha),(\Omega^\#(\beta),\beta)]_{\mathcal{L}^*}$ in (2) as*

$$[(\Omega^\#(\alpha),\alpha),(\Omega^\#(\beta),\beta)]_{\mathcal{L}^*} = d_*(\Omega(\alpha,\beta)) + \Omega^\#([\alpha,\beta]_*).$$

Proof. Consider the expressions:

$$\begin{aligned}
&\langle(\mathcal{L}^*_\alpha\Omega^\#(\beta) - \mathcal{L}^*_\beta\Omega^\#(\alpha) + d_*(\Omega(\alpha,\beta))),\gamma\rangle \qquad \gamma\in A^*\\
&\langle\mathcal{L}^*_\alpha\Omega^\#(\beta),\gamma\rangle = \rho_*(\alpha)\Omega(\beta,\gamma) - \Omega(\beta,[\alpha,\gamma]_*)\\
&\langle\mathcal{L}^*_\beta\Omega^\#(\alpha),\gamma\rangle = \rho_*(\beta)\Omega(\alpha,\gamma) - \Omega(\alpha,[\beta,\gamma]_*)\\
&\langle d_*(\Omega(\alpha,\beta)),\gamma\rangle = \rho_*(\gamma)\Omega(\alpha,\beta).
\end{aligned}$$

With this, we can write

$$\begin{aligned}
&\langle[(\Omega^\#(\alpha),\alpha),(\Omega^\#(\beta),\beta)]_{\mathcal{L}^*},\gamma\rangle\\
&\quad = \langle(\mathcal{L}^*_\alpha\Omega^\#(\beta) - \mathcal{L}^*_\beta\Omega^\#(\alpha) + d_*(\Omega(\alpha,\beta))),\gamma\rangle\\
&\quad = \rho_*(\alpha)\Omega(\beta,\gamma) - \rho_*(\beta)\Omega(\alpha,\gamma) + \rho_*(\gamma)\Omega(\alpha,\beta)\\
&\qquad - \Omega(\beta,[\alpha,\gamma]_*) - \Omega([\beta,\gamma]_*,\alpha) = d_*\Omega(\alpha,\beta,\gamma) + \Omega([\alpha,\beta]_*,\gamma)\\
&\quad = \langle d_*\Omega(\alpha,\beta) + \Omega^\#([\alpha,\beta]_*),\gamma\rangle.
\end{aligned}$$

□

5.2.1. Two elements of the graph. Consider $\alpha, \beta \in I^{\perp}$ and let us study the deformed bracket of two elements $(\Omega^{\#}(\alpha), \alpha)$ and $(\Omega^{\#}(\beta), \beta)$. We want to study the conditions for it to be closed, so as for D to be closed with respect to $[\cdot,\cdot]_{\diamond}^{\mathcal{N}}$. We have to consider the four different terms which appears in (43):

$$[(\Omega^{\#}(\alpha), \alpha), (\Omega^{\#}(\beta), \beta)]_{\diamond}^{\mathcal{N}} = ([\Omega^{\#}(\alpha), \Omega^{\#}(\beta)]_N + [(\Omega^{\#}(\alpha), \alpha), (\Omega^{\#}(\beta), \beta)]_{\mathcal{L}_{\mathcal{D}_*}^{\Upsilon}}, \\ [((\Omega^{\#}(\alpha), \alpha), (\Omega^{\#}(\beta), \beta)]_{\mathcal{L}_{\mathcal{D}}^{N}} + [\alpha, \beta]_{\Upsilon}) \tag{47}$$

We will analyse now each one separately, and collect the conditions arising from each one.

- *First term of (47)*: As the deformation defined by $\mathcal{N}$ on $A \oplus A^*$ is assumed to be trivial, we know that

  $$[\Omega^{\#}(\alpha), \Omega^{\#}(\beta)]_N = [N\Omega^{\#}(\alpha), \Omega^{\#}(\beta)] + [\Omega^{\#}(\alpha), N\Omega^{\#}(\beta)] - N[\Omega^{\#}(\alpha), \Omega^{\#}(\beta)]$$

  and

  $$N[\Omega^{\#}(\alpha), \Omega^{\#}(\beta)]_N = [N\Omega^{\#}(\alpha), N\Omega^{\#}(\beta)].$$

  And we have from condition (42)

  $$[\Omega^{\#}(\alpha), \Omega^{\#}(\beta)]_N = [\Omega^{\#}(\Upsilon\alpha), \Omega^{\#}(\beta)] + [\Omega^{\#}(\alpha), \Omega^{\#}(\Upsilon\beta)] - N[\Omega^{\#}(\alpha), \Omega^{\#}(\beta)].$$

  Now we use the relation between these brackets and the corresponding Nijenhuis brackets to write

  $$[\Omega^{\#}(\Upsilon\alpha), \Omega^{\#}(\beta)] = \Omega^{\#}([\Upsilon\alpha, \beta]_{\Omega}) + \tfrac{1}{2}[\Omega, \Omega](\Upsilon\alpha, \beta) \\ [\Omega^{\#}(\alpha), \Omega^{\#}(\Upsilon\beta)] = \Omega^{\#}([\alpha, \Upsilon\beta]_{\Omega}) + \tfrac{1}{2}[\Omega, \Omega](\alpha, \Upsilon\beta) \\ N[\Omega^{\#}(\alpha), \Omega^{\#}(\beta)] = N\left(\Omega^{\#}([\alpha, \beta]_{\Omega}) + \tfrac{1}{2}[\Omega, \Omega](\alpha, \beta)\right).$$

- *For the second term of (47)*:

  $$[(\Omega^{\#}(\alpha), \alpha), (\Omega^{\#}(\beta), \beta)]_{\mathcal{L}_{\mathcal{D}_*}^{\Upsilon}} = [(N\Omega^{\#}(\alpha), \Upsilon\alpha), (\Omega^{\#}(\beta), \beta)]_{\mathcal{L}^*} \\ + [(\Omega^{\#}(\alpha), \alpha), (N\Omega^{\#}(\beta), \Upsilon\beta)]_{\mathcal{L}^*} - N[(\Omega^{\#}(\alpha), \alpha), (\Omega^{\#}(\beta), \beta)]_{\mathcal{L}^*}.$$

  Our intention now is to use lemma 5.2 to write the conditions on the brackets in terms of the tensor Ω and the operators N and Υ. In order to do that, we can use again the compatibility condition (42) and write

  $$[(\Omega^{\#}(\alpha), \alpha), (\Omega^{\#}(\beta), \beta)]_{\mathcal{L}_{\mathcal{D}_*}^{\Upsilon}} = [(\Omega^{\#}(\Upsilon\alpha), \Upsilon\alpha), (\Omega^{\#}(\beta), \beta)]_{\mathcal{L}^*} \\ + [(\Omega^{\#}(\alpha), \alpha), (\Omega^{\#}(\Upsilon\beta), \Upsilon\beta)]_{\mathcal{L}^*} - N[(\Omega^{\#}(\alpha), \alpha), (\Omega^{\#}(\beta), \beta)]_{\mathcal{L}^*}.$$

  These terms are of the form of lemma 5.2, and therefore, we can write

  $$[(\Omega^{\#}(\alpha), \alpha), (\Omega^{\#}(\beta), \beta)]_{\mathcal{L}_{\mathcal{D}_*}^{\Upsilon}} = d_*\Omega(\Upsilon\alpha, \beta) + d_*\Omega(\alpha, \Upsilon\beta) \\ - N(d_*\Omega(\alpha, \beta)) + \Omega^{\#}([\Upsilon\alpha, \beta] + [\alpha, \Upsilon\beta] - \Upsilon[\alpha, \beta]).$$

- *The third term of (47)*:

  $$[(\Omega^{\#}(\alpha), \alpha), (\Omega^{\#}(\beta), \beta)]_{\mathcal{L}_{\mathcal{D}}^{N}} = [(N\Omega^{\#}(\alpha), \Upsilon\alpha), (\Omega^{\#}(\beta), \beta)]_{\mathcal{L}} \\ + [(\Omega^{\#}(\alpha), \alpha), (N\Omega^{\#}(\beta), \Upsilon\beta)]_{\mathcal{L}} - \Upsilon[(\Omega^{\#}(\alpha), \alpha), (\Omega^{\#}(\beta), \beta)]_{\mathcal{L}}.$$

  Once again, we can use condition (42) and write

  $$[(\Omega^{\#}(\alpha), \alpha), (\Omega^{\#}(\beta), \beta)]_{\mathcal{L}_{\mathcal{D}}^{N}} = [(\Omega^{\#}(\Upsilon\alpha), \Upsilon\alpha), (\Omega^{\#}(\beta), \beta)]_{\mathcal{L}} \\ + [(\Omega^{\#}(\alpha), \alpha), (\Omega^{\#}(\Upsilon\beta), \Upsilon\beta)]_{\mathcal{L}} - \Upsilon[(\Omega^{\#}(\alpha), \alpha), (\Omega^{\#}(\beta), \beta)]_{\mathcal{L}}.$$

Hence we have, defining

$$[\alpha,\beta]_\Omega = \mathcal{L}_{\Omega^\#(\alpha)}\beta - \mathcal{L}_{\Omega^\#(\beta)}\alpha - d\Omega(\alpha,\beta),$$

that we can rewrite the expression above as

$$[(\Omega^\#(\alpha),\alpha),(\Omega^\#(\beta),\beta)]_{\mathcal{L}_\mathcal{D}^N} = [\alpha,\Upsilon\beta]_\Omega + [\Upsilon\alpha,\beta]_\Omega - \Upsilon[\alpha,\beta]_\Omega.$$

- *The final term of (47)*: this one follows directly from (14). The expression reads

$$[\alpha,\beta]_\Upsilon = [\Upsilon\alpha,\beta] + [\alpha,\Upsilon\beta] - \Upsilon[\alpha,\beta]. \tag{48}$$

We can summarize now the above results: consider a deformation $\mathcal{N} = N \times \Upsilon$ of the Courant algebroid structure as above. Then, the product $[(\Omega^\#(\alpha),\alpha),(\Omega^\#(\beta),\beta)]_\diamond^\mathcal{N}$ turns out to be (Y,Ψ) where

$$\begin{aligned} Y = {} & d_*\Omega(\Upsilon\alpha,\beta) + \tfrac{1}{2}[\Omega,\Omega](\Upsilon\alpha,\beta) + d_*\Omega(\alpha,\Upsilon\beta) + \tfrac{1}{2}[\Omega,\Omega](\alpha,\Upsilon\beta) - N(d_*\Omega(\alpha,\beta) \\ & + \tfrac{1}{2}[\Omega,\Omega](\alpha,\beta)) + \Omega^\#([\Upsilon\alpha,\beta] + [\alpha,\Upsilon\beta] - \Upsilon[\alpha,\beta]) + \Omega^\#([\Upsilon\alpha,\beta]_\Omega \\ & + [\alpha,\Upsilon\beta]_\Omega - \Upsilon[\alpha,\beta]_\Omega) \end{aligned} \tag{49}$$

and

$$\Psi = [\alpha,\beta]_\Upsilon + [\alpha,\Upsilon\beta]_\Omega + [\Upsilon\alpha,\beta]_\Omega - \Upsilon[\alpha,\beta]_\Omega. \tag{50}$$

We must ask the point (Y,Ψ) to belong to D. But that is equivalent to asking that

$$\begin{aligned} & d_*\Omega(\Upsilon\alpha,\beta) + \tfrac{1}{2}[\Omega,\Omega](\Upsilon\alpha,\beta) + d_*\Omega(\alpha,\Upsilon\beta) + \tfrac{1}{2}[\Omega,\Omega](\alpha,\Upsilon\beta) \\ & \quad - N(d_*\Omega(\alpha,\beta) + \tfrac{1}{2}[\Omega,\Omega](\alpha,\beta)) \in I. \end{aligned} \tag{51}$$

But that condition is granted by the fact that D is a Dirac structure and the compatibility condition (42) holds. Hence, we can write for this case:

Lemma 5.3. *Consider a Dirac structure $D = (I,\Omega)$ and a deformation of $A \oplus A^*$ in the form $\mathcal{N} = N \times \Upsilon : A \oplus A^* \to A \oplus A^*$, where N and Υ are deformations of the Lie algebroids A and A^* such that they define a trivial deformation of the Lie bialgebroid (A, A^*) and of the double $A \oplus A^*$. Assume that $\mathcal{N}$ and Ω satisfy condition (42). Then, the bracket $[(\Omega^\#(\alpha),\alpha),(\Omega^\#(\beta),\beta)]_\diamond^\mathcal{N}$ is closed in D for all $\alpha,\beta \in I^\perp$.*

5.2.2. One element of I and one element of the graph of $\Omega^\#|_{I^\perp}$. We consider the bracket of an element of I with an element of the graph.

$$\begin{aligned} [(X,0),(\Omega^\#(\alpha),\alpha)]_\diamond^\mathcal{N} = {} & [(NX,0),(\Omega^\#(\alpha),\alpha)]_\diamond + [(X,0),(N\Omega^\#(\alpha),\Upsilon\alpha)]_\diamond \\ & - N \times \Upsilon[(X,0),(\Omega^\#(\alpha),\alpha)]_\diamond. \end{aligned} \tag{52}$$

If we assume the compatibility condition (42), we can write

$$\begin{aligned} [(X,0),(\Omega^\#(\alpha),\alpha)]_\diamond^\mathcal{N} = {} & [(NX,0),(\Omega^\#(\alpha),\alpha)]_\diamond + [(X,0),(\Omega^\#(\Upsilon\alpha),\Upsilon\alpha)]_\diamond \\ & - N \times \Upsilon[(X,0),(\Omega^\#(\alpha),\alpha)]_\diamond. \end{aligned} \tag{53}$$

For the analysis of each term, we can use Liu's construction in [12] and write

$$[(X,0),(\Omega^\#(\alpha),\alpha)]_\diamond = \left([X,\Omega^\#(\alpha)] - \mathcal{L}^*_\alpha X - \Omega^\#(\mathcal{L}_X\alpha) + \Omega^\#(\mathcal{L}_X\alpha), \mathcal{L}_X\alpha\right). \tag{54}$$

For the first term of (52), we have

$$[(NX,0),(\Omega^\#(\alpha),\alpha)]_\diamond = ([NX,\Omega^\#(\alpha)] - \mathcal{L}^*_\alpha(NX) - \Omega^\#(\mathcal{L}_{NX}\alpha) + \Omega^\#(\mathcal{L}_{NX}\alpha), \mathcal{L}_{NX}\alpha).$$

We can rewrite the second term on the right-hand side of (52) as

$$\begin{aligned} [(X,0),\Omega^\#(\Upsilon\alpha),\Upsilon\alpha)]_\diamond = {} & ([X,\Omega^\#(\Upsilon\alpha)] \\ & - \mathcal{L}^*_{\Upsilon\alpha}X - \Omega^\#(\mathcal{L}_X(\Upsilon\alpha)) + \Omega^\#(\mathcal{L}_X(\Upsilon\alpha)), \mathcal{L}_X(\Upsilon\alpha)). \end{aligned}$$

And finally, the third term of (52) is simply written as

$$N \times \Upsilon[(X, 0), (\Omega^{\#}(\alpha), \alpha)]_{\diamond} = (N([X, \Omega^{\#}(\alpha)] - \mathcal{L}^*_{\alpha} X + \Omega^{\#}(\mathcal{L}_X \alpha) - \Omega^{\#}(\mathcal{L}_X \alpha)), \Upsilon \mathcal{L}_X \alpha).$$

Proceeding as Liu in [12], we can try to relate the condition arising from these equations with a bracket in A^*. It is evident that the bracket $[(X, 0), (\Omega^{\#}(\alpha), \alpha)]^{\mathcal{N}}_{\diamond}$ belongs to D, if

$$\begin{aligned}
&[NX, \Omega^{\#}(\alpha)] + [X, \Omega^{\#}(\Upsilon\alpha)] - N[X, \Omega^{\#}(\alpha)] - \mathcal{L}^*_{\alpha} NX - \mathcal{L}^*_{\Upsilon\alpha} X + N\mathcal{L}^*_{\alpha} X \\
&\qquad - \Omega^{\#}(\mathcal{L}_{NX}\alpha + \mathcal{L}_X(\Upsilon\alpha) - \Upsilon(\mathcal{L}_X\alpha)) \in I.
\end{aligned} \tag{55}$$

Now, for the terms in the first line of (55), we use the fact that, for $Y \in I$ and $\beta, \gamma \in I^{\perp}$,

$$\langle [Y, \Omega^{\#}(\gamma)], \beta\rangle = -\rho(\Omega^{\#}(\gamma))\langle Y, \beta\rangle + \langle Y, \mathcal{L}_{\Omega^{\#}(\gamma)}\beta\rangle = \langle Y, \mathcal{L}_{\Omega^{\#}(\gamma)}\beta\rangle.$$

Then, for the second line of (55), we will use the same relation for the dual elements

$$\langle \mathcal{L}^*_{\gamma} Y, \beta\rangle = \rho_*(\gamma)\langle Y, \beta\rangle - \langle Y, [\gamma, \beta]_*\rangle = -\langle Y, [\gamma, \beta]_*\rangle.$$

Finally, we require some relation to deal with the elements of the third line of (55):

$$\begin{aligned}
\langle Y, \mathcal{L}_{\Omega^{\#}(\gamma)}\beta\rangle + \rho(Y)\Omega(\beta, \gamma) &= i_Y \mathcal{L}_{\Omega^{\#}(\gamma)}\beta + i_Y d(\Omega(\beta, \gamma)) \\
&= i_Y i_{\Omega^{\#}(\gamma)} d\beta + i_Y d i_{\Omega^{\#}(\gamma)}\beta - i_Y d i_{\Omega^{\#}(\gamma)}\beta \\
&= i_Y i_{\Omega^{\#}(\gamma)} d\beta = -i_{\Omega^{\#}(\gamma)} i_Y d\beta \\
&= -i_{\Omega^{\#}(\gamma)} \mathcal{L}_Y \beta = i_{\Omega^{\#}(\mathcal{L}_Y \beta)}\gamma = \langle \Omega^{\#}(\mathcal{L}_Y \beta), \gamma\rangle
\end{aligned}$$

where we used the definition of the Lie derivative and the antisymmetry of Ω.

Now we can write that

$$\begin{aligned}
&\langle [NX, \Omega^{\#}(\alpha)] + [X, \Omega^{\#}(\Upsilon\alpha)] - N[X, \Omega^{\#}(\alpha)], \beta\rangle \\
&\qquad = \langle X, N^*\mathcal{L}_{\Omega^{\#}(\alpha)}\beta + \mathcal{L}_{\Omega^{\#}(\Upsilon\alpha)}\beta - \mathcal{L}_{\Omega^{\#}(\alpha)} N^*\beta\rangle \\
&-\langle \mathcal{L}^*_{\alpha} NX + \mathcal{L}^*_{\Upsilon\alpha} X - N\mathcal{L}^*_{\alpha} X, \beta\rangle = \langle X, N^*[\alpha, \beta]_* + [\Upsilon\alpha, \beta]_* - [\alpha, N^*\beta]_*\rangle \\
&-\langle \Omega^{\#}(\mathcal{L}_{NX}\alpha + \mathcal{L}_X\Upsilon\alpha - \Upsilon(\mathcal{L}_X\alpha)), \beta\rangle = -\langle X, N^*\mathcal{L}_{\Omega^{\#}(\beta)}\alpha + N^* d(\Omega(\alpha, \beta))\rangle \\
&\qquad - \langle X, \mathcal{L}_{\Omega^{\#}(\beta)}(\Upsilon\alpha) + d(\Omega(\Upsilon\alpha, \beta))\rangle + \langle X, \mathcal{L}_{\Omega^{\#}(N^*\beta)}\alpha + d(\Omega(\alpha, N^*\beta))\rangle.
\end{aligned}$$

We must remember now that the compatibility condition in this case is (42). With this, we can write that (55) is equivalent to

$$[\Upsilon\alpha, \beta]_* - [\alpha, N^*\beta]_* + N^*[\alpha, \beta]_* + [\Upsilon\alpha, \beta]_{\Omega} - [\alpha, N^*\beta]_{\Omega} + N^*[\alpha, \beta]_{\Omega} \in I^{\perp}. \tag{56}$$

But again, this condition is granted by the fact that D is a Dirac structure. Therefore we prove

Lemma 5.4. *Consider a Dirac structure $D = (I, \Omega)$ and a deformation of $A \oplus A^*$ in the form $\mathcal{N} = N \times \Upsilon : A \oplus A^* \to A \oplus A^*$, where N and Υ define trivial deformations of the Lie algebroids A and A^* such that they define a trivial deformation of the Lie bialgebroid (A, A^*). Assume that conditions (42) are satisfied. Then, the bracket $[(X, 0), (\Omega^{\#}(\alpha), \alpha)]^{\mathcal{N}}_{\diamond}$ is closed in D.*

5.2.3. Two elements of I. This condition is trivial to obtain, just asking the subbundle I to be stable under the action of the deformation:

Lemma 5.5. *Consider a Dirac structure $D = (I, \Omega)$ and a deformation of $A \oplus A^*$ in the form $\mathcal{N} = N \times \Upsilon : A \oplus A^* \to A \oplus A^*$, where N and Υ define trivial deformations of the Lie algebroids A and A^* and such that they define a trivial deformation of the Lie bialgebroid*

(A, A^*). Then, the bracket $[(X,0),(Y,0)]^{\mathcal{N}}_{\diamond}$ is closed in D if the bundle I is closed under the deformation N, i.e.

$$[X, Y]_N \in I \qquad \forall X, Y \in I. \tag{57}$$

Summarizing, we obtain that

Proposition 5.1. *Let (A, A^*) be a Lie bialgebroid over a manifold M and D a Dirac structure represented by a characteristic pair (I, Ω). Consider an operator $\mathcal{N} = N \times \Upsilon : A \oplus A^* \to A \oplus A^*$ and assume*

- *N (resp. Υ) are trivial deformations of the Lie algebroid A (resp. A^*).*
- *$\mathcal{N}$ is compatible with Ω in the sense $N\Omega^\# = \Omega^\#\Upsilon$.*
- *$\Upsilon(I^\perp) \subset I^\perp$ and $N(I) \subset I$.*
- *$N^* + \Upsilon \in \mathbb{R}$.*
- *$N^2 \in \mathbb{R}$.*

Then, $\mathcal{N}$ defines a trivial deformation of the Lie algebroid structure of D.

6. Deformation of Dirac structures II

Consider again a Lie bialgebroid (A, A^*) and a Dirac structure $D \subset A \oplus A^*$ represented by the characteristic pair (I, Ω). We know from the first section that the subbundle D is endowed with a Lie algebroid structure, whose Lie bracket corresponds to the skew-symmetric operation (2) which is closed and satisfies the Jacobi identity on the sections of D. From that point of view, we can study a deformation of that Lie algebroid structure *per se*, without considering the deformation of the double of the corresponding Lie bialgebroid. In spite of this, we will, for simplicity, consider the transformations in terms of the elements of the double, though restricted to the subbundle D. Therefore, we will define now the transformation $\mathcal{N} : A \oplus A^* \to A \oplus A^*$ as

$$\mathcal{N}(X, \alpha) = (NX, \Upsilon\alpha). \tag{58}$$

As before, we are going to consider that the operators $N : A \to A$ and $\Upsilon : A^* \to A^*$ are Nijenhuis operators for the Lie algebroids A and A^*.

Note 6.1. *Of course, more general transformations are also possible, introducing two more tensors $J_\alpha : A^* \to A$ and $J_X : A \to A^*$, but we will not consider that case in this paper. We plan to study the deformations arising from that type of situation in the future.*

First of all, we must ask $\mathcal{N}$ to preserve D as a bundle. We have, in a completely analogous way to the previous section:

Lemma 6.1. *Consider a deformation $\mathcal{N} = N \times \Upsilon$ of the double of a Lie bialgebroid (A, A^*) and a Dirac structure $D \subset A \oplus A^*$ with characteristic pair (I, Ω). Then, $\mathcal{N}$ preserves the subbundle $I \oplus \mathrm{graph}(\Omega)|_{I^\perp}$ if and only if the bivector Ω and the operators satisfy*

$$N\Omega^\# = \Omega^\# \circ \Upsilon \qquad N(I) \subset I \qquad \Upsilon(I^\perp) \subset I^\perp. \tag{59}$$

If the transformation satisfies these conditions the bundle is preserved. Now, we must ask the transformation to be trivial with respect to the Lie bracket of D. To do that, we have to ask the Nijenhuis torsion of $\mathcal{N}$ to vanish.

We can do that in two steps, as above, by defining a deformed bracket:

$$
\begin{aligned}
[(X_\alpha + \Omega^\#(\alpha), \alpha), (X_\beta + \Omega^\#(\beta), \beta)]_\diamond^{\mathcal{N}} = \big(&[X_\alpha + \Omega^\#(\alpha), X_\beta + \Omega^\#(\beta)]_N \\
&+ [(X_\alpha + \Omega^\#(\alpha), \alpha), (X_\beta + \Omega^\#(\beta), \beta)]_{\mathcal{L}_{\mathcal{D}_*}^\Upsilon}, \\
&[(X_\alpha + \Omega^\#(\alpha), \alpha), (X_\beta + \Omega^\#(\beta), \beta)]_{\mathcal{L}_{\mathcal{D}}^N} + [\alpha, \beta]_\Upsilon\big)
\end{aligned} \tag{60}
$$

where now the diagonal factors are

$$
\begin{aligned}
[(X_1, \alpha_1), (X_2, \alpha_2)]_{\mathcal{L}_{\mathcal{D}}^N} = {}& [\mathcal{N}(X_1, \alpha_1), (X_2, \alpha_2)]_{\mathcal{L}} + [(X_1, \alpha_1), \mathcal{N}(X_2, \alpha_2)]_{\mathcal{L}} \\
&- \mathcal{N}[(X_1, \alpha_1), (X_2, \alpha_2)]_{\mathcal{L}}
\end{aligned} \tag{61}
$$

and

$$
\begin{aligned}
[(X_1, \alpha_1), (X_2, \alpha_2)]_{\mathcal{L}_{\mathcal{D}_*}^\Upsilon} = {}& [\mathcal{N}(X_1, \alpha_1), (X_2, \alpha_2)]_{\mathcal{L}^*} + [(X_1, \alpha_1), \mathcal{N}(X_2, \alpha_2)]_{\mathcal{L}^*} \\
&- \mathcal{N}[(X_1, \alpha_1), (X_2, \alpha_2)]_{\mathcal{L}^*}
\end{aligned} \tag{62}
$$

where the brackets $[\cdot,\cdot]_N$ and $[\cdot,\cdot]_\Upsilon$ are the brackets of the Lie algebroids $(A, [\cdot,\cdot]_N, \hat{N})$ and $(A^*, [\cdot,\cdot]_\Upsilon, \hat{\Upsilon})$ defined in (14).

The first condition to ask to this bracket is that it is closed on D. If it is closed, it is trivial to see that a vanishing Nijenhuis torsion is equivalent to the condition:

$$
\mathcal{N}[(X_\alpha + \Omega^\#(\alpha), \alpha), (X_\beta + \Omega^\#(\beta), \beta)]_\diamond^{\mathcal{N}} = [\mathcal{N}(X_\alpha + \Omega^\#(\alpha), \alpha), \mathcal{N}(X_\beta + \Omega^\#(\beta), \beta)]_\diamond. \tag{63}
$$

Now, in the undeformed case, closedness of the bracket (2) in ΓD is equivalent to the Lie condition because, being a Courant algebroid, the Jacobi identity on the double of the Lie bialgebroid (A, A^*) is equal to the exterior derivative of the closedness condition. This is no longer true in the deformed case if we do not consider a deformation of the double of the Lie bialgebroid. Hence, the closedness condition is not enough for the deformed bracket to be a Lie bracket. The Jacobi condition must be verified. In the general case, this is a very lengthy task, since the number of terms is very high. For the sake of simplicity, we will consider now a simpler case, which is the following:

Lemma 6.2. *Consider a Lie bialgebroid (A, A^*) a Dirac structure D with characteristic pair (I, Ω) and the bracket (2). Consider a deformation $\mathcal{N} : A \oplus A^* \to A \oplus A^*$ which preserves D (i.e. $\mathcal{N}(D) \subset D$) and with an associated bracket (60) that is closed in ΓD and satisfies condition (63). If the operator $\mathcal{N}$ is invertible, then the bracket (60) is of Lie type.*

Proof. Skewsymmetry and closedness are granted by the assumptions. Condition (63) allows us to write:

$$
[(X_\alpha + \Omega^\#(\alpha), \alpha), (X_\beta + \Omega^\#(\beta), \beta)]_\diamond^{\mathcal{N}} = \mathcal{N}^{-1}[\mathcal{N}(X_\alpha + \Omega^\#(\alpha), \alpha), \mathcal{N}(X_\beta + \Omega^\#(\beta), \beta)]_\diamond.
$$

Hence, the expression of the Jacobi identity for the deformed bracket takes the form

$$
\begin{aligned}
[[e_1, e_2]_\diamond^{\mathcal{N}}, e_3]_\diamond^{\mathcal{N}} + [[e_3, e_1]_\diamond^{\mathcal{N}}, e_2]_\diamond^{\mathcal{N}} + [[e_2, e_3]_\diamond^{\mathcal{N}}, e_1]_\diamond^{\mathcal{N}} = {}& \mathcal{N}^{-1}([[\mathcal{N}e_1, \mathcal{N}e_2]_\diamond, \mathcal{N}e_3]_\diamond \\
&+ [[\mathcal{N}e_3, \mathcal{N}e_1]_\diamond, \mathcal{N}e_2]_\diamond + [[\mathcal{N}e_2, \mathcal{N}e_3]_\diamond, \mathcal{N}e_1]_\diamond)
\end{aligned} \tag{64}
$$

which vanish if $\mathcal{N}(D) \subset D$ because the undeformed bracket is Lie. □

To consider the conditions to be satisfied by the deformation to ensure the closedness condition, we must proceed as in the previous section.

6.1. The closedness condition

The closedness condition is completely analogous to the previous case, since we did the computation without asking the transformation to be trivial for the double $A \oplus A^*$. We see then that the compatibility condition (59) ensures the closedness of the operation:

Lemma 6.3. *Consider a deformation $\mathcal{N} = N \times \Upsilon$ defined on $D \subset A \oplus A^*$ where (A, A^*) is a Lie bialgebroid and D a Dirac structure with the characteristic pair (I, Ω). Assume that*

- *$\mathcal{N}$ is invertible.*
- *N (resp. Υ) are trivial deformations of the Lie algebroid A (resp. A^*).*
- *$\mathcal{N}$ is compatible with Ω in the sense $N\Omega^\# = \Omega^\#\Upsilon$.*
- *$\Upsilon(I^\perp) \subset I^\perp$ and $N(I) \subset I$.*

Then, the bracket (60) is closed in ΓD.

6.2. The triviality condition

From here comes the main difference with the previous case. As we saw above, once that we know that the deformed bracket is closed in D, the condition for the deformation to be trivial reduces to studying if

$$\mathcal{N}[(X_\alpha + \Omega^\#(\alpha), \alpha), (X_\beta + \Omega^\#(\beta), \beta)]_\diamond^{\mathcal{N}} = [\mathcal{N}(X_\alpha + \Omega^\#(\alpha), \alpha), \mathcal{N}(X_\beta + \Omega^\#(\beta), \beta)]_\diamond.$$

Once more, we are going to proceed by analysing the three different types of brackets that we can find above.

6.2.1. Two elements of the graph. We must study for which symmetric deformation in the form $\mathcal{N}$ above the following relation holds:

$$\mathcal{N}[(\Omega^\#(\alpha), \alpha), (\Omega^\#(\beta), \beta)]_\diamond^{\mathcal{N}} = [\mathcal{N}(\Omega^\#(\alpha), \alpha), \mathcal{N}(\Omega^\#(\beta), \beta)]_\diamond.$$

From the previous analysis, we know that the above expression can be written as

$$\mathcal{N}[(\Omega^\#(\alpha), \alpha), (\Omega^\#(\beta), \beta)]_\diamond^{\mathcal{N}} = (NZ_s, \Upsilon\Psi_s)$$

where

$$\begin{aligned} NZ_s &= NZ_I + N\Omega^\#([\Upsilon\alpha, \beta] + [\alpha, \Upsilon\beta] - \Upsilon[\alpha, \beta]) + N\Omega^\#([\alpha, \Upsilon\beta]_\Omega + [\Upsilon\alpha, \beta]_\Omega - \Upsilon[\alpha, \beta]_\Omega) \\ &= NZ_I + \Omega^\#(\Upsilon[\Upsilon\alpha, \beta] + \Upsilon[\alpha, \Upsilon\beta] - \Upsilon^2[\alpha, \beta]) \\ &\quad + \Omega^\#(\Upsilon[\alpha, \Upsilon\beta]_\Omega + \Upsilon[\Upsilon\alpha, \beta]_\Omega - \Upsilon^2[\alpha, \beta]_\Omega) \end{aligned}$$

and

$$\Upsilon\Psi_s = \Upsilon[\Upsilon\alpha, \beta] + \Upsilon[\alpha, \Upsilon\beta] - \Upsilon^2[\alpha, \beta] + \Upsilon[\alpha, \Upsilon\beta]_\Omega + \Upsilon[\Upsilon\alpha, \beta]_\Omega - \Upsilon^2[\alpha, \beta]_\Omega$$

where

$$\begin{aligned} Z_I = d_*\Omega(\Upsilon\alpha, \beta) &+ \tfrac{1}{2}[\Omega, \Omega](\Upsilon\alpha, \beta) + d_*\Omega(\alpha, \Upsilon\beta) \\ &+ \tfrac{1}{2}[\Omega, \Omega](\alpha, \Upsilon\beta) - N(d_*\Omega(\alpha, \beta) + \tfrac{1}{2}[\Omega, \Omega](\alpha, \beta)). \end{aligned}$$

As N^* and Υ are assumed to be sum-compatible Nijenhuis operators for the Lie algebroid A^*, the corresponding Nijenhuis torsion vanishes, and hence we can write

$$NZ_s = NZ_I + \Omega^\#([\Upsilon\alpha, \Upsilon\beta]) + \Omega^\#(\Upsilon[\alpha, \Upsilon\beta]_\Omega + \Upsilon[\Upsilon\alpha, \beta]_\Omega - \Upsilon^2[\alpha, \beta]_\Omega)$$

and

$$\Upsilon\Psi_s = [\Upsilon\alpha, \Upsilon\beta] + \Upsilon[\alpha, \Upsilon\beta]_\Omega + \Upsilon[\Upsilon\alpha, \beta]_\Omega - \Upsilon^2[\alpha, \beta]_\Omega.$$

On the other hand

$$[\mathcal{N}(\Omega^{\#}(\alpha), \alpha), \mathcal{N}(\Omega^{\#}(\beta), \beta)]_{\diamond} = (\hat{Z}, \hat{\Psi})$$

where

$$\hat{Z} = d_*\Omega(\Upsilon\alpha, \Upsilon\beta) + \tfrac{1}{2}[\Omega, \Omega](\Upsilon\alpha, \Upsilon\beta) + \Omega^{\#}([\Upsilon\alpha, \Upsilon\beta] + [\Upsilon\alpha, \Upsilon\beta]_{\Omega})$$

and

$$\hat{\Psi} = [\Upsilon\alpha, \Upsilon\beta] + [\Upsilon\alpha, \Upsilon\beta]_{\Omega}.$$

Therefore, we conclude that

Lemma 6.4. *The deformation driven by $\mathcal{N}$ is trivial on* $\mathrm{graph}(\Omega)|_{I^{\perp}}$ *if*

$$NZ_I = d_*\Omega(\Upsilon\alpha, \Upsilon\beta) + \tfrac{1}{2}[\Omega, \Omega](\Upsilon\alpha, \Upsilon\beta)$$

and

$$\Upsilon[\alpha, \Upsilon\beta]_{\Omega} + \Upsilon[\Upsilon\alpha, \beta]_{\Omega} - \Upsilon^2[\alpha, \beta]_{\Omega} = [\Upsilon\alpha, \Upsilon\beta]_{\Omega}.$$

6.2.2. An element of the graph and an element of I. We have to study then which of these deformations are trivial, i.e. to check whether

$$\mathcal{N}[(X, 0), (\Omega^{\#}(\alpha), \alpha)]_{\diamond}^{\mathcal{N}} = [\mathcal{N}(X, 0), \mathcal{N}(\Omega^{\#}(\alpha), \alpha)]_{\diamond}$$

i.e.

$$N[X, \Omega^{\#}(\alpha)]_N - N\mathcal{L}^*_{\alpha}(NX) - N\mathcal{L}^*_{\Upsilon\alpha}X + N^2\mathcal{L}^*_{\alpha}X = [NX, N\Omega^{\#}(\alpha)] - \mathcal{L}^*_{\Upsilon\alpha}NX \tag{65}$$

for the sections of A and

$$\Upsilon\mathcal{L}_{NX}\alpha + \Upsilon\mathcal{L}_X(\Upsilon\alpha) - \Upsilon^2\mathcal{L}_X\alpha = \mathcal{L}_{NX}(\Upsilon\alpha) \tag{66}$$

for the sections of A^*.

In the above expression, the fact that N is a Nijenhuis operator for A ensures that

$$N[X, \Omega^{\#}(\alpha)]_N = [NX, N\Omega^{\#}(\alpha)].$$

Then, we see that the deformation is trivial on these elements if

$$\begin{cases} N\mathcal{L}^*_{\alpha}(NX) + N\mathcal{L}^*_{\Upsilon\alpha}X - N^2\mathcal{L}^*_{\alpha}X = \mathcal{L}^*_{\Upsilon\alpha}NX \\ \Upsilon\mathcal{L}_{NX}\alpha + \Upsilon\mathcal{L}_X(\Upsilon\alpha) - \Upsilon^2\mathcal{L}_X\alpha = \mathcal{L}_{NX}(\Upsilon\alpha). \end{cases} \tag{67}$$

6.2.3. Two elements of I. In this case the triviality condition is granted by N defining a trivial deformation of A. As I is a closed subbundle of A, the triviality follows.

Summarizing, we obtain the following result:

Proposition 6.1. *Consider a Lie bialgebroid (A, A^*) a Dirac structure D with characteristic pair (I, Ω) and the bracket (2). Consider a deformation $\mathcal{N} : A \oplus A^* \to A \oplus A^*$ which preserves D (i.e. $\mathcal{N}(D) \subset D$) and with an associated bracket (60) that is closed in ΓD. Assume that $\mathcal{N}$ is invertible and satisfies*

- *the conditions of lemma 6.4 and*
- *condition (67).*

Then, $\mathcal{N}$ is a Nijenhuis operator for the Lie algebroid structure of D.

7. Definition and examples of Dirac–Nijenhuis manifolds

At this point, we can finally introduce the concept of Dirac–Nijenhuis manifolds, as a generalization of the Poisson–Nijenhuis case, but with some differences. Let us analyse carefully the situation in the Poisson case. As we saw above, a Poisson–Nijenhuis manifold is endowed with a Poisson tensor Λ such that there is a compatible Nijenhuis operator N for M such that

- The tensor Λ is also a Poisson tensor for the deformed Lie structure $[\cdot,\cdot]_N$.
- The tensor $N\Lambda$ is Poisson for the undeformed Lie structure.

If we look at this situation from a Dirac perspective, we realize

- The first point implies that the Dirac bundle graph$(\Lambda^{\#}) \subset TM \oplus T^*M$ is also a Dirac structure when we consider the deformed skew-symmetric operation associated with the deformation.
- The second point implies that there exists a Dirac structure for the bundle graph$(N\Lambda^{\#}) \subset TM \oplus T^*M$. In general, this bundle will be different from the previous one.

These are the properties that are adapted to the Dirac case in [14], but only for the case of the trivial Lie bialgebroid structure of $TM \oplus T^*M$, i.e. with the null structure on the cotangent bundle. Hence, they consider a deformation of the double of that Lie bialgebroid, and the effect that such a mapping has on Dirac structures defined on them. In the general case, the original Dirac structure $L \subset TM \oplus T^*M$ is transformed into a new one $L_1 \subset TM \oplus T^*M$, as in the Poisson–Nijenhuis case (where we have two different Poisson tensors). Therefore there are two different Lie structures on L and on L_1.

Our approach is somehow different, since we are interested only in the effect that the deformation has on the original bundle $D \subset A \oplus A^*$ (for (A, A^*) a general Lie bialgebroid). From our point of view, the most natural way of studying a deformation of a Dirac structure D is to study the change in the Lie structure of D (hence on its Lie algebroid structure) under transformations which map D on itself, and not on different regions of the double of the Lie bialgebroid which define new bundles. It is only in that case that it makes sense to talk about Nijenhuis operators and trivial deformations, since only in that case is the bracket of a section of D and of a section of the transformed bundle well defined in D (this type of bracket is required in the definition of the Nijenhuis torsion). As a conclusion, only in certain cases can the results of [14] be compared with ours, namely those where $L = L_1$. A similar thing happens in [1], particularly for the simple examples they study. While the definition which is provided is, formally, equivalent to ours, the examples studied reflect an approach quite different. The main point is that they define a deformation of the Leibniz algebra which defines the Courant algebroid structure, and, when restricted to the Dirac bundle, it defines a new closed product for it. That is indeed true. But, from our point of view, though the construction makes perfect sense as a product, such a deformation cannot be considered to be a trivial deformation of the Lie algebroid structure since the image of the Dirac bundle is not contained in it. The resulting product is indeed contained, but for instance, the factors in the Nijenhuis torsion tensor are not. The main difference with our framework is that the approach based on the deformation of Leibniz algebras, being more natural to study the deformation of the Courant algebroid structure, is less natural to study the deformation of the Lie algebroid structure of D than ours, since then, from our point of view, it makes more sense to study, directly, the deformations of the Lie structure. In that case the Nijenhuis torsion (with respect to the Lie bracket, which is the operation that is being deformed) is defined on the Dirac structure, and on it, it identically vanishes. As a result, it is not possible to compare our examples with those in [1], since they refer to two different frameworks.

Hence we focus on the deformation of the Lie algebroid structure of a given Dirac structure D defined on the double of a Lie bialgebroid (A, A^*). We can consider two different settings:

- A transformation that defines a trivial deformation of the skew-symmetric operation (2) on the double of the Lie bialgebroid and defines a new Courant algebroid structure on it, and, as a consequence, transforms the Lie algebroid structure of D into a new one.
- A transformation that deforms the skew-symmetric operation (2) in such a way that it defines a trivial deformation of the Lie algebroid structure of D. In this case, we do not care about the transformation of the Courant algebroid structure of the double.

Therefore, we define

Definition 7.1. *Let M be a differentiable manifold and (A, A^*) be a Lie bialgebroid defined on it. Then, $(D, \mathcal{N})$ is said to be a* Dirac–Nijenhuis structure of type I *(or M to be a* Dirac–Nijenhuis manifold of type I*) if D is a Dirac structure for the Lie bialgebroid (A, A^*) and the operator $\mathcal{N} : A \oplus A^* \to A \oplus A^*$ preserves D (i.e. $\mathcal{N}(D) = D$) and defines*

- *a trivial deformation of the skew-symmetric algebra of $A \oplus A^*$ whose Nijenhuis torsion vanishes (with respect to the skew-symmetric bracket defined on $A \oplus A^*$) and*
- *a new skew-symmetric operation $[\cdot,\cdot]_{\mathcal{N}}$ for which D is also a Dirac structure.*

This corresponds to the most natural definition of the deformation: we deform the double of the Lie bialgebroid $A \oplus A^*$ and, at the same time, we deform the Lie algebroid structure of the Dirac bundle $D \subset A \oplus A^*$. The conditions for D to be a Dirac structure in the deformed structure can also be read from theorem 2.1, by using the deformed exterior derivatives and Schouten brackets.

But we also saw that we have the possibility of deforming the Lie algebroid structure of D without paying attention to the structure of the double. Hence, it makes sense to define

Definition 7.2. *Let M be a differentiable manifold and (A, A^*) be a Lie bialgebroid defined on it. Then, $(D, \mathcal{N})$ is said to be a* Dirac–Nijenhuis structure of type II *(or M to be a* Dirac–Nijenhuis manifold of type II*) if D is a Dirac structure with respect to the Lie bialgebroid (A, A^*), and the operator $\mathcal{N} : A \oplus A^* \to A \oplus A^*$ defines a trivial deformation of the Lie algebroid structure of D, i.e. the Nijenhuis torsion of $\mathcal{N}$ with respect to the bracket (2) vanishes identically on ΓD. Of course, this also implies that the transformation $\mathcal{N}$ preserves D.*

It is important to note that the difference between the definitions does not affect the Dirac structure itself, it just refers to the relation of it with the double of the Lie bialgebroid where it is defined. Hence, it makes sense to call both cases *Dirac–Nijenhuis structures*.

In any case, the results of the previous sections lead to

Theorem 7.1. *Let (A, A^*) be a Lie bialgebroid over a manifold M and D be a Dirac structure represented by a characteristic pair (I, Ω). Consider an operator $\mathcal{N} = N \times \Upsilon : A \oplus A^* \to A \oplus A^*$ and assume*

- *N (resp. Υ) is a Nijenhuis operator of the Lie algebroid A (resp. A^*).*
- *$\mathcal{N} = N \times \Upsilon$ is a Nijenhuis operator of $A \oplus A^*$.*
- *N and Υ are compatible with D in the sense that they preserve the bundle, i.e. $\Upsilon(I^\perp) \subset I^\perp$ and $N(I) \subset I$ and besides*
$$N\Omega^\# = \Omega^\#\Upsilon.$$
- *I is closed under the bracket $[\cdot,\cdot]_N$. Equivalently, we can claim that the structure $(I, [\cdot,\cdot]_N, N \circ \rho)$ is a Lie subalgebroid of A.*

Then, M is a Dirac–Nijenhuis manifold of type I.

Theorem 7.2. *Let (A, A^*) be a Lie bialgebroid over a manifold M and D be a Dirac structure represented by a characteristic pair (I, Ω). Consider an operator $\mathcal{N} = N \times \Upsilon : A \oplus A^* \to A \oplus A^*$ and assume*

- *N (resp. Υ) is a Nijenhuis operator of the Lie algebroid A (resp. A^*).*
- *N and Υ are compatible with D in the sense that they preserve the bundle, i.e. $\Upsilon(I^\perp) \subset I^\perp$ and $N(I) \subset I$ and besides*

$$N\Omega^\# = \Omega^\#\Upsilon.$$

- *$\mathcal{N}$ is invertible and satisfies*
  - *the conditions of lemma 6.4.*
  - *condition (67).*
- *I is closed under the bracket $[\cdot,\cdot]_N$. Equivalently, we can claim that the structure $(I, [\cdot,\cdot]_N, N \circ \rho)$ is a Lie subalgebroid of A.*

Then, M is a Dirac–Nijenhuis manifold of type II.

7.1. Application: the Dirac structure associated with a Poisson–Nijenhuis manifold

As a simple application, let us see how the concept of a Dirac–Nijenhuis manifold contains the Poisson–Nijenhuis as a particular case. Consider then (M, Λ, N) to be a Poisson–Nijenhuis manifold, and let us verify that we can prove it to be a Dirac–Nijenhuis one. In order to do that, starting from the natural Lie bialgebroid structure in (TM, T^*M), we must consider a set of sum-compatible deformations of the corresponding structures, which lead to a pair of sum-compatible deformations of the double $TM \oplus T^*M$. The natural choice, from the results we saw in section 3.3, is to consider the deformation studied by Kosmann-Schwarzbach in [10] and combine it with the identity transformation.

7.1.1. Can we define a trivial deformation for the double? In short, the answer is yes, we can. But if it is done with an operator N which is compatible with the Poisson tensor, then it cannot preserve the graph of Λ, at least, with the type of deformations of doubles of Lie bialgebroids that we have considered in this paper. Let us look at that point carefully. Consider the double $TM \oplus T^*M$ and a deformation of the Lie bialgebroid

$$(N, \Upsilon) : (TM, T^*M) \to (TM, T^*M).$$

If the deformation of the double must be of the form $\mathcal{N} = N \times \Upsilon$, we saw that a sufficient condition for it to be trivial is $N + \Upsilon^* \in \mathbb{R}$.

As we know that given a trivial deformation of the Lie algebroid A, $N : A \to A$, it is always sum-compatible with the identity, clearly $\lambda + N$ is also a deformation for A for any $\lambda \in \mathbb{R}$. It is trivial to see that $\lambda - N$ also defines a trivial deformation.

Consider now the case of a Poisson–Nijenhuis structure. We know from [10] that the deformation (N, Id) defines a deformation of the Lie bialgebroid structure of (TM, T^*M), for N an arbitrary Nijenhuis operator of the tangent bundle TM. But we also know that the dual operator $N^* : T^*M \to T^*M$ also defines a Nijenhuis operator for the Lie bialgebroid structure defined on T^*M by the Poisson tensor. Therefore, the results on compatibility of deformations above imply that $\lambda + N$ is a Nijenhuis operator for TM and $\lambda - N^*$ is a Nijenhuis operator for T^*M. We can claim then that, for this case and, in general, for any pair of dual Lie algebroids:

Lemma 7.1. *Consider a Lie bialgebroid (A, A^*). Let $N : A \to A$ be a Lie algebroid A and assume that N^* is also a Nijenhuis operator for A^*. Then, if $N^2 \in \mathbb{R}$ it is possible to define a trivial deformation of the sum $A \oplus A^*$ defined as*

$$\mathcal{N} = (\lambda + N) \times (\lambda - N^*). \tag{68}$$

This lemma ensures that, for the case of a Poisson–Nijenhuis manifold, we can define a deformation of the double structure of $TM \oplus T^*M$. But it is important to note that, if we deform the double by using a Nijenhuis operator N from TM, the graph of the Poisson tensor Λ cannot be preserved since that would imply that, as we saw in lemma 6.3,

$$(\lambda + N)\Lambda^\# = \Lambda^\#(\lambda - N^*) \Rightarrow N\Lambda^\# = -\Lambda^\# N^*.$$

And we know that the condition for (M, Λ, N) to be a Poisson–Nijenhuis manifold is that $N\Lambda^\# = \Lambda^\# N^*$. Hence, it is impossible to satisfy also the compatibility condition of the Dirac bundle graph(Λ).

7.1.2. The deformation of graph(Λ). We cannot use the above results to classify a Poisson–Nijenhuis manifold as a Dirac–Nijenhuis of type I: we can define a deformation of the double of the Lie bialgebroid structure of (TM, T^*M), but we cannot use that deformation to define a deformation of the Dirac structure defined by the graph of the Poisson tensor. It is still possible, though, to classify them as a Dirac–Nijenhuis structure of type II, as we are going to see.

The procedure is then to forget about the double structure of $TM \oplus T^*M$ and study directly a deformation of the Dirac structure of graph(Λ). We use theorem 7.2 for this: assume then that we have an invertible Nijenhuis operator N on M, and that we consider the standard Lie bialgebroid structure of (TM, T^*M). Then, as we know that the compatibility condition $N\Lambda^\# = \Lambda^\# N^*$ is satisfied, we just have to check the relations arising from the triviality condition. But in this case, we know that $I = 0$, $d_*\Lambda = [\Lambda, \Lambda] = 0$ and $[\Lambda, \Lambda] = 0$. This ensures that all the conditions are satisfied. Therefore, we prove

Proposition 7.1. *Let (M, N, Λ) be a Poisson–Nijenhuis manifold such that N is invertible. Then,* (graph(Λ), $N \times N^*$) *is a Dirac–Nijenhuis manifold of type II.*

7.2. Deformations of Kähler manifolds and Dirac–Nijenhuis manifolds of type I

Let us finally consider an example of a Dirac–Nijenhuis manifold of type I. Consider a Kähler manifold (M, J, ω), where J is the complex structure and ω is the symplectic form.

We can consider several different situations. For the Lie bialgebroid structure, we can consider at least two possibilities:

- The natural structure on the tangent bundle and the null one on T^*M.
- The natural structure on the TM and the one associated with the Poisson tensor on the cotangent bundle.

Now, the complex structure J satisfies the condition $J^2 \in \mathbb{R}$ required for the triviality of a deformation of the double $TM \oplus T^*M$. We can also see that the other condition is also satisfied, since the fact of being a Kähler manifold implies that, denoting by g the Kähler metric and by X, Y two arbitrary vector fields,

$$\begin{aligned}\langle Y, J^*\omega^\#(X)\rangle &= \omega(X, JY) = g(JX, JY) = g(X, Y) = -g(J^2X, Y)\\ &= -\omega(JX, Y) = -\langle Y, \omega^\# J(X)\rangle\end{aligned}$$

where we used the relation between the Kähler metric and symplectic form given by $\omega(X, Y) = g(JX, Y)$ and the expression of the dual operator for the complex structure $J^* : T^*M \to T^*M$. But rewriting this relation in terms of the Poisson tensor associated with the symplectic form $P^\# = \omega^{-1\#}$, we obtain

$$JP^\# = -P^\# J^*.$$

Therefore, if we consider a deformation such as $\mathcal{N} = (\lambda + J) \times (\lambda - J^*)$:

- It defines a trivial deformation of the double $TM \oplus T^*M$ because the conditions of theorem 4.1 are satisfied.
- It defines a trivial deformation of the Dirac structure defined by the graph of the tensor P, i.e. $D = \mathrm{graph}(P)$, because the conditions of proposition 5.1 are also satisfied.
- It defines a trivial deformation of any Dirac structure defined as $D = I \oplus \mathrm{graph}(P)$ for I any subbundle which is
  - closed for the undeformed product (2).
  - stable under the action of J, i.e. $J(I) \subset I$.
  - closed for the deformed product of A, i.e. $[\cdot,\cdot]_J$.

Summarizing, we have proved

Proposition 7.2. *Let (M, J, ω) be a Kähler manifold and let us denote by P the Poisson tensor corresponding to the symplectic structure ω. Assume that the tensor J defines a trivial deformation of the Lie structure of the tangent bundle TM. Then, any Dirac structure defined on M as $D = I \oplus \mathrm{graph}(P)$ for I any subbundle which is*

- *closed for the undeformed product (2),*
- *stable under the action of J, i.e. $J(I) \subset I$ and*
- *closed for the deformed product of A, i.e. $[\cdot,\cdot]_J$,*

defines a Dirac–Nijenhuis structure of type I as $(D, (\lambda - J) \times (\lambda + J))$.

8. Conclusions and future work

We have presented some results concerning the deformation of different geometric structures, namely Lie algebroids, Lie bialgebroids, the double of Lie bialgebroids (and Courant algebroids) and, as our main objective, Dirac structures. The treatment of the deformations of the double of Lie bialgebroids is partial, as only the effect that a deformation on the Lie bialgebroid has on the double has been studied, while more general approaches are also possible. Nonetheless, they are rich enough to allow a large variety of deformations of the Dirac structures defined on the doubles, while being much simpler from the computational point of view. Hence, we consider that the choice is justified.

In the future we intend to consider those main cases not studied in the present paper, and study more general deformations of Lie bialgebroids and Courant algebroids. For this, we will certainly explore the lines presented in [1], since they offer a very powerful framework in precisely these aspects.

References

[1] Cariñena J F, Grabowski J and Marmo G 2004 Courant algebroids and Lie bialgebroids contractions *Technical Report* (*Preprint* math.DG/0402020)

[2] Courant T J 1990 Dirac manifolds *Trans. Am. Math. Soc.* **319** 631–61

[3] Courant T J and Weinstein A 1988 Beyond Poisson structures *Séminaire Sud-Rhodanien de géométrie, Travaux en cours* vol 27 (New York: Hermann) pp 39–49

[4] Dorfman I 1987 Dirac structures of integrable evolution equations *Phys. Lett.* A **125** 240–6

[5] Dorfman I 1993 *Dirac Structures and Integrability of Nonlinear Evolution Equation* (New York: Wiley)

[6] Grabowski J and Marmo G 2003 The graded Jacobi algebras and (co)homology *J. Phys. A: Math. Gen.* **36** 161–8

[7] Grabowski J and Urbanski P 1997 Lie algebroids and Poisson–Nijenhuis structures *Rep. Math. Phys.* **40** 195–208

[8] Iglesias-Ponte D and Marrero J C 2002 Generalized Lie bialgebroids and strong Jacobi–Nijenhuis structures *Extracta Math.* **17** 259–71

[9] Kosmann-Schwarzbach Y 1995 Exact Gernstenhaber algebras and Lie bialgebroids *Acta Appl. Math.* **41** 153–65

[10] Kosmann-Schwarzbach Y 1996 The Lie bialgebroid of a Poisson–Nijenhuis manifold *Lett. Math. Phys.* **38** 421–8

[11] Kosmann-Schwarzbach Y and Magri F 1990 Poisson–Nijenhuis structures *Ann. Inst. H Poincaré Phys. Theor.* **53** 35–81

[12] Liu Z-J 2000 Some remarks on Dirac structures and Poisson reductions *Poisson Geometry* vol 51 (Warsaw: Banach Center Publications) pp 165–73

[13] Liu Z J, Weinstein A and Xu P 1997 Manin triples for Lie bialgebroids *J. Diff. Geom.* **45** 547–74

[14] He L and Liu B 2004 Dirac–Nijenhuis manifolds *Rep. Math. Phys.* **53** 123–42

[15] Mackenzie K M and Xu P 1994 Lie bialgebroids and Poisson groupoids *Duke Math. J.* **73** 415–52

[16] Magri F and Morosi C 1984 A geometrical characterization of integrable Hamiltonian systems through the theory of Poisson–Nijenhuis manifolds *Technical Report* University of Milano

[17] Maschke B and van der Schaft A J 1992 Port controlled Hamiltonian systems: modelling origins and system-theoretic properties *Proc. 2nd IFAC NOLCOS (Bordeaux)* pp 282–8

[18] Nunes da Costa J M 2002 A characterization of strict Jacobi–Nijenhuis manifolds through the theory of Lie algebroids *Rep. Math. Phys.* **50** 339–47

[19] Nunes da Costa J M and Clemente-Gallardo J 2004 Dirac structures for generalized Lie bialgebroids *J. Phys. A: Math. Gen.* **37** 2671–92

[20] Pradines J 1966 Théorie de Lie pour les groupoïdes différentiables. Relations entre propriétés locales et globales *C. R. Acad. Sci. Paris* I **263** 907–10

[21] Roytenberg D 1999 Courant algebroids, derived brackets and even symplectic supermanifolds *PhD Thesis* UC Berkeley

[22] van der Schaft A J and Dalsmo M 1999 On representations and integrability of mathematical structures in energy-conserving physical systems *SIAM J. Control Optim.* **37** 54–91